Homework problem set 5 Math 423/502

1. Let $x \in R$ be an element which is not a zero-divisor. Show that the first Tor group is the module of "x-torsion in M", namely:

$$\mathrm{Tor}_1(M, R/(x)) = \{m \in M \mid xm = 0\}\,.$$

Also compute $\mathrm{Tor}_1(R/(x), M)$ directly from the definition and see that it is also isomorphic to $\{m \in M \mid xm = 0\}$. (It is true that $\mathrm{Tor}_k(A, B) \cong \mathrm{Tor}_k(B, A)$ in general, but we have not yet proved that).

2. Let I and J be ideals of R. Let IJ be the ideal generated by elements ab where $a \in I$ and $b \in B$. Show that

$$\mathrm{Tor}_1(R/I, R/J) = (I \cap J)/(IJ)$$

and use this to show that $I \cap J = IJ$ if $I + J = R$.

3. Let $x \in R$ be an element which is not a zero-divisor. Compute

$$\mathrm{Ext}^i_R(R/(x), M)$$

and in particular compute $\mathrm{Ext}^i(\mathbb{Z}/n, \mathbb{Z}/m)$ for all $n, m \in \mathbb{Z}$.

4. Provide an example of non-isomorphic extensions

$$\begin{aligned} 0 \to B \to &X \ \to A \to 0 \\ 0 \to B \to &X' \to A \to 0 \end{aligned}$$

such that $X \cong X'$.

5. Let $R = \mathbb{C}[x, y]$ and let $\mathbb{C}_0 = R/(x, y)$. Use a free resolution of $\mathbb{C}_0$ to compute $\mathrm{Ext}^1_R(\mathbb{C}_0, \mathbb{C}_0)$. Explicitely write the extensions parameterized by $\mathrm{Ext}^1_R(\mathbb{C}_0, \mathbb{C}_0)$.

6. (**Optional**) Let I be any ideal in a ring R. Prove that

$$\mathrm{Ext}^1(R/I, R/I) = \mathrm{Hom}(I/I^2, R/I) = \mathrm{Hom}(\mathrm{Tor}_1(R/I, R/I), R/I).$$

(Use problem 2).