Homological Algebra

Feb 19, 2019

$\mathbb{Z}$-modules a.k.a Abelian gps and vector-spaces are important ~~examples~~ special cases

R - commutative ring with unit.

<u>Def'n</u> A <u>complex</u> of R-modules is a sequence

$$F_\bullet = \left[\cdots \to F_{i+1} \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} F_{i-1} \to \cdots \right]$$

of R-module maps such that $d_i \circ d_{i+1} = 0$ $\quad (d^2 = 0)$

Index set is $\mathbb{Z}$ but we often only consider cases where $F_i = 0$ unless $i \geq 0$ or $i \leq 0$ or $i \in [a,b]$.

<u>Def'n</u> The homology of $F_\bullet$ at F_i is

$$H_i(F_\bullet) = \ker d_i / \operatorname{Im} d_{i+1}$$

Homological algebra is the study of complexes and their homology. As we will see, this is often a tool to study other things of more fundamental interest. (modules themselves, systems of eq's and their solutions, functors applied to modules, topology of spaces).

Jargon: • F_i is the ~~piece~~ term of degree i

• the maps d_i are called boundary generators, or differentials (we will see examples to help explain this).

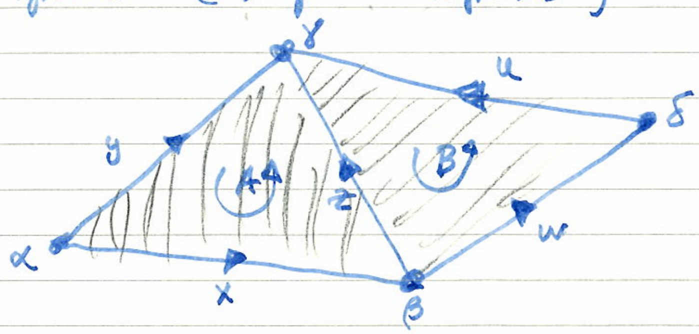
• Elements f such that $df = 0$ are called "cycles" or "closed", Elements f such that $f = dg$ some g are ~~stat~~ called "boundaries" or "exact". All boundaries are cycles.

• If $H_i(F_\bullet) = 0$ we say $F_\bullet$ is exact at F_i

If $H_i(F_\bullet) = 0 \ \forall i$ we say $F_\bullet$ is an exact sequence

$H_i(F_\bullet)$ measures "cycles which are not boundaries".

examples Historically this arose studying triangulations of spaces (simplicial complexes)



$$0 \to \mathbb{Z}A \oplus \mathbb{Z}B \xrightarrow{\partial} \mathbb{Z}x \oplus \mathbb{Z}y \oplus \mathbb{Z}z \oplus \mathbb{Z}w \oplus \mathbb{Z}u \xrightarrow{\partial} \mathbb{Z}\alpha \oplus \mathbb{Z}\beta \oplus \mathbb{Z}\gamma \oplus \mathbb{Z}\delta \to 0$$

degree 2 generated by dim 2 simplices

degree 1 dim 1 simplices

deg 0 dim 0 simplices

$\partial A = x + z - y \quad \partial B = w + u - z \quad \partial x = \beta - \alpha \quad \partial u = \gamma - \delta$
$\partial z = \gamma - \beta \quad \partial w = \delta - \beta$
$\partial y = \gamma - \alpha$

- is $x + w + u - y$ closed? is it a boundary?

what is the homology of this complex?

<u>Example 2</u> Let $U \subset \mathbb{R}^3$ be some open set

deg ~~3~~   deg 2   deg 1   deg 0

$$C^\infty(U) \xrightarrow[\text{grad}]{} \mathrm{Vect}(U) \xrightarrow[\text{curl}]{} \mathrm{Vect}(U) \xrightarrow[\text{div}]{} C^\infty(U)$$

$\mathrm{curl}(\mathrm{grad}\, f) = 0 \qquad \mathrm{div}(\mathrm{curl}\, \vec{F}) = 0$

A vector field $\vec{F}$ is conservative if $\vec{F} = \mathrm{grad} f$.
$\mathrm{curl}\, \vec{F} = 0$ is a necessary condition
but depending on U, not all $\mathrm{curl}\, \vec{F} = 0$ vector fields are conservative. H_2 of this complex ~~measure~~ is the "space" of irrotational but non-conservative vector fields.

The above is better expressed in the language of differential forms

eg. $f \longmapsto df = f_x\, dx + f_y\, dy + f_z\, dz$

$$P\, dx dy + Q\, dy dz + R\, dz dx \longmapsto (P_z + Q_x + R_y)\, dx\, dy\, dz$$

this context is the origin of the words "differential" and "exact".

(Optional) More examples? Compute the homology of $0 \to \mathbb{Z} \xrightarrow{\cdot 0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 0} \mathbb{Z} \to 0$

degree: $\quad 3 \quad 2 \quad 1 \quad 0$

$H_3 = \mathbb{Z} \quad H_2 = 0 \quad H_1 = \mathbb{Z}/2 \quad H_0 = \mathbb{Z}$

what about $0 \to \mathbb{Z}/2 \xrightarrow{\cdot 0} \mathbb{Z}/2 \xrightarrow{\cdot 2 = \cdot 0} \mathbb{Z}/2 \xrightarrow{\cdot 0} \mathbb{Z}/2 \to 0$?

Two simplices  glued to each other along boundary gives

$$0 \to \mathbb{Z}^2 \xrightarrow[d_2]{\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}} \mathbb{Z}^3 \xrightarrow[d_1]{\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}} \mathbb{Z}^3 \to 0$$

what's the homology? is d_1 onto? what is a quantity which is 0 on the image of d_1? what's the $\ker d_2$? what's the $\ker d_1$?

commutative algebra
~~For us~~, homological algebra gives us ways of studying arbitrary R-modules in terms of "nice" modules, for example free modules.

<u>Example</u> Complexes arise in the study of systems of linear equations whose coef's lie in some ring R.

S'pose we wish to describe the set of solutions to a system of n_0 linear equations in n_1 variables. This system defines a module map

$$\phi : \underset{\substack{\| \\ F_1}}{R^{\oplus n_1}} \longrightarrow \underset{\substack{\| \\ F_0}}{R^{\oplus n_0}}$$

A vector $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_{n_1} \end{pmatrix}$ is a solution if $\phi x = 0$

i.e. $x \in \ker \phi$. Describing solutions might consist of finding some set of solutions $\vec{x}_1, \cdots, \vec{x}_{n_2}$ so that every solution is a linear combination of $\vec{x}_1, \cdots, \vec{x}_{n_2}$

i.e. some $F_2 = R^{\oplus n_2}$ and map

$$F_2 \longrightarrow F_1 \xrightarrow{\phi} F_0$$

such that the above is a complex and exact at F_1.

If R were a field, we could demand that the solutions were <u>independent</u> i.e. $\ker(F_2 \to F_1) = 0$

for example, let $R = \mathbb{C}[a,b,c]$ polynomial ring

and suppose our linear system of equations is 1 equation

3 variables:

$$aX_1 + bX_2 + cX_3 = 0$$

$$R^{\oplus 3} \xrightarrow{(a,b,c)} R$$

Some solutions: ~~(0, -c, b)~~ $\begin{pmatrix} 0 \\ -c \\ b \end{pmatrix} \begin{pmatrix} c \\ 0 \\ -a \end{pmatrix} \begin{pmatrix} -b \\ a \\ 0 \end{pmatrix}$

$$R^{\oplus 3} \xrightarrow{\begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}} R^{\oplus 3} \xrightarrow{(a,b,c)} R$$

These solutions generate kernel, but no 2 generate.

there is one relation: $a\begin{pmatrix} 0 \\ -c \\ b \end{pmatrix} + b\begin{pmatrix} c \\ 0 \\ -a \end{pmatrix} + c\begin{pmatrix} -b \\ a \\ 0 \end{pmatrix} = 0$

$$0 \to R \xrightarrow[\begin{pmatrix} a \\ b \\ c \end{pmatrix}]{} R^{\oplus 3} \xrightarrow[\begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}]{} R^{\oplus 3} \xrightarrow[(a,b,c)]{} R$$

no further relations (relations among the relations) in this case.

Can view the above as studying the (non-free) module $\mathbb{C}[a,b,c]/(a,b,c)$. Note that the homology of the above complex is 0 except at the right most spot where it is $R/(a,b,c)$.

<u>Def'n</u> Let M be a R-module. $F_\bullet = [\cdots \to F_i \to F_{i-1} \to \cdots \to F_0]$ is a <u>free resolution</u> of M if $F_\bullet$ is a complex

$H_i(F_\bullet) = 0$ except $i=0$ $H_0(F_\bullet) = M$,

$F_i = R^{\oplus a_i}$ is a free module.

These always exist: let $e_1, \cdots, e_n$ be some set of generators for M and let $F_0 \cong R^{\oplus n} \xrightarrow{\phi} M$ be the map that sends $\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ to e_i

then $M \cong F_0 / \ker\phi$ since $\ker\phi$ is an R-module we can find generators and find a surjective map

$F_1 \twoheadrightarrow \ker\phi$

we get $F_1 \to F_0$ with $M = F_0 / \mathrm{im}(F_1 \to F_0)$

do this iteratively with the kernels.

<u>example</u> Let $R = \mathbb{C}[x]/(x^n)$ Let $M = R / x^m R$

$0 < m < n$.

The map $\begin{matrix} 1 \longmapsto 1 \\ R \twoheadrightarrow M \end{matrix}$

has kernel generated by x^m so consider

$$R \xrightarrow{\cdot x^m} R \twoheadrightarrow M$$

the kernel of $R \xrightarrow{\cdot x^m} R$ is generated by

x^{n-m} so $R \xrightarrow{x^{n-m}} R \xrightarrow{\cdot x^m} R$ is the next step

and we see that the semi-infinite complex

$$F_\bullet = \left[\cdots \longrightarrow R \xrightarrow{\cdot x^m} R \xrightarrow{\cdot x^{n-m}} R \xrightarrow{\cdot x^m} R \to 0 \right]$$

is a free resolution of M.

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Free resolutions are far from unique. We will want to understand when exactly 2 different complexes ~~give~~ are resolutions of the same module. More generally, we want a way to compare 2 different complexes:

Let $\underset{(F_\bullet, \phi_\bullet)}{F_\bullet}$ & $\underset{(G_\bullet, \psi_\bullet)}{G_\bullet}$ be complexes.

<u>Def'n</u> A <u>map of complexes</u> $\alpha_\bullet : F_\bullet \to G_\bullet$ is a sequence of maps α_i such that

$$\begin{array}{ccccc} \cdots \longrightarrow & F_i & \xrightarrow{\phi_i} & F_{i-1} & \to \cdots \\ & \downarrow{\alpha_i} & & \downarrow{\alpha_{i-1}} & \\ \longrightarrow & G_i & \xrightarrow{\psi_i} & G_{i-1} & \to \cdots \end{array}$$

commutes

$$\alpha_{i-1} \circ \phi_i = \psi_i \circ \alpha_i$$

A map of complexes induces a map in homology:

$$\alpha_i : H_i(F_\bullet) \longrightarrow H_i(G_\bullet)$$

$$\ker \phi_i \xrightarrow{\alpha_i} G_i$$

- image of α_i here lands in $\ker \psi_i$
- it takes $\operatorname{Im} \phi_{i+1}$ to $\operatorname{Im} \psi_{i+1}$

] α_i well defined in homology.

When do two maps of complexes $\alpha_\circ : F_\circ \to G_\circ$ $\beta_\circ : F_\circ \to G_\circ$ induce the same map in homology?

Hard question in general, but there is a nice sufficient condition which, in some important special cases, is also necessary.

<u>Def'n.</u> We say $\alpha_\circ, \beta_\circ : F_\circ \to G_\circ$ are <u>homotopic</u> $\alpha_\circ \simeq \beta_\circ$ if there exists a <u>chain homotopy</u>, i.e. maps $h_i : F_i \to G_{i+1}$ such that $\alpha_i - \beta_i = \psi_{i+1} h_i + h_{i-1} \phi_i$

$$\begin{tikzcd} F_{i+1} \arrow[r, "\phi_{i+1}"] \arrow[d] & F_i \arrow[r, "\phi_i"] \arrow[d, "\alpha_i - \beta_i"] \arrow[dl, "h_i"'] & F_{i-1} \arrow[d] \arrow[dl, "h_{i-1}"'] \\ G_{i+1} \arrow[r, "\psi_{i+1}"'] & G_i \arrow[r] & G_{i-1} \end{tikzcd}$$

clear sufficient for $\alpha_\circ - \beta_\circ = 0 : H_\circ(F_\circ) \to H_\circ(G_\circ)$:

$$F_i \longrightarrow G_i$$

$$x \longmapsto (\alpha_i - \beta_i)x = \underbrace{\psi_{i+1} h_i x}_{\substack{0 \text{ since its} \\ \text{in the image of } \psi_{i+1}}} + h_{i-1} \overbrace{\phi_i x}^{0 \text{ since } x \in \ker \phi_i}$$

For free resolutions, this is especially nice:

<u>Prop.</u> Let $F_\bullet = [\to \cdots \to F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0]$ and $G_\bullet = [\to \cdots \to G_1 \xrightarrow{\psi_1} G_0]$ be free resolutions of modules M & N. Then every module map $M \xrightarrow{\beta} N$ is induced by a map of complexes $\alpha_\bullet : F_\bullet \to G_\bullet$ and α is determined by β upto homotopy.

This proposition says that the category of R-modules could be replaced by a category of free resolutions with maps upto homotopy. This foreshadows some sophisticated aspects of the subject: derived categories and derived functors.

For now, think of this as "we can study R-modules by studying their free resolutions up to homotopy".

The proof is also a good illustration of diagram chasing.

$$\begin{tikzcd}
\cdots \arrow[r] & F_1 \arrow[r, "\phi_1"] \arrow[d, dashed, "\alpha_1"] \arrow[dr, "\alpha_0\circ\phi_1"] & F_0 \arrow[r, "\phi_0"] \arrow[d, dashed, "\alpha_0"] \arrow[dr, "\beta\circ\phi_0"] & M \arrow[r] \arrow[d, "\beta"] & 0 \\
\arrow[r] & G_1 \arrow[r, "\psi_1"] & G_0 \arrow[r, "\psi_0"] & N \arrow[r] & 0
\end{tikzcd}$$

α_0 exists: since ψ_0 is surjective, for each generator $e_i \in F_0$ there is some elt. in G_0 in $\psi_0^{-1}(\beta(\phi_0(e_i)))$ that we have $\alpha_0(e_i)$ equal then $\operatorname{Im}(\alpha_0 \circ \phi_1) \subset \operatorname{Ker}(\psi_0) = \operatorname{Im}\psi_1$ so we can also find the lift α_1 in the same way. Proceed inductively.

Now suppose α_0 & α_0' are two different cx maps $F_0 \to G_0$ fitting into the above diagram (and hence both inducing β in homology). We which to show $\alpha_0 \simeq \alpha_0'$, i.e. $\alpha_0 - \alpha_0'$ homotopic to 0.

by subtracting the maps in the above diagram we see that this is equivalent to showing that any map α_0 inducing 0 in coh. is chain homotopic to 0.

$$\begin{tikzcd}
F_2 \arrow[r, "\phi_2"] \arrow[d] & F_1 \arrow[r, "\phi_1"] \arrow[d, "\alpha_1"] \arrow[dl, dashed, "h_1"'] & F_0 \arrow[r, "\phi_0"] \arrow[d, "\alpha_0"] \arrow[dl, dashed, "h_0"'] & M \arrow[d, "0"] \\
G_2 \arrow[r, "\psi_2"'] & G_1 \arrow[r, "\psi_1"'] & G_0 \arrow[r, "\psi_0"'] & N
\end{tikzcd}$$

we want $\alpha_i = h_{i-1}\phi_i + \psi_{i+1}h_i$

h_0 exists: $\operatorname{Im}\alpha_0 \subset \operatorname{Ker}\psi_0 = \operatorname{Im}\psi_1$ so we can lift.

$$\psi_1(h_0\phi_1 - \alpha_1) = \alpha_0\phi_1 - \psi_1\alpha_1 = 0$$

so $\operatorname{Im}(h_0\phi_1 - \alpha_1) \subset \operatorname{Ker}\psi_1 = \operatorname{Im}\psi_2$

so $\exists\, h_1$ s.t. $\psi_2 h_1 = h_0\phi_1 - \alpha_1$

adjusting the sign of h_1 we get $\alpha_1 = h_0\phi_1 + \psi_2 h_1$

proceed by induction $\blacksquare$.

Every $\mathbb{Q}$-module is free

Projective $\mathbb{Z}$-modules are free.

In general, not all proj. R-modules are free

The ideal $(2, 1+\sqrt{-5}) \subset \mathbb{Z}[\sqrt{-5}]$ is proj. not free des:

The only ~~thing~~ way we used ~~about~~ the freeness of the modules in the free resolution was a lifting property:

<u>Def'n</u> A module P is <u>projective</u> if for every surjective map of modules $\alpha: M \twoheadrightarrow N$ and map $\beta: P \to N$ there exists a lift $\gamma: P \to M$ s.t. $\beta = \alpha\gamma$

$$\begin{tikzcd} & P \arrow[d, "\beta"] \arrow[dl, dotted, "\exists\gamma"'] \\ M \arrow[r, two heads, "\alpha"'] & N \end{tikzcd}$$

the previous result is true for projective resolution slightly more general than free resolutions).

- There is also a dual notion of <u>injective module</u> and injective resolution. We will return to this later.

Let $F'_\bullet$, $F_\bullet$, $F''_\bullet$ be complexes

<u>Def'n</u> A short exact sequence of cxs is a sequence of maps of cxs

$$0 \to F'_\bullet \xrightarrow{\alpha_\bullet} F_\bullet \xrightarrow{\beta_\bullet} F''_\bullet \to 0$$

such that $0 \to F'_k \xrightarrow{\alpha_k} F_k \xrightarrow{\beta_k} F''_k \to 0$ is short exact for all k.

we get induced maps in homology and a <u>connecting homomorphism</u> $\delta_\bullet$:

$$H_k(F'_\bullet) \xrightarrow{\alpha_k} H_k(F_\bullet) \xrightarrow{\beta_k} H_k(F''_\bullet) \xrightarrow{\delta_k} H_{k-1}(F'_\bullet)$$

$$\begin{tikzcd}
0 \arrow[r] & F'_{k+1} \arrow[r] \arrow[d] & F_{k+1} \arrow[r] \arrow[d] & F''_{k+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F'_{k} \arrow[r] \arrow[d] & F_{k} \arrow[r, "\beta"] \arrow[d] & F''_{k} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F'_{k-1} \arrow[r] \arrow[d] & F_{k-1} \arrow[r] \arrow[d] & F''_{k-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F'_{k-2} \arrow[r] & F_{k-2} \arrow[r] & F''_{k-2} \arrow[r] & 0
\end{tikzcd}$$

$y \mapsto x$, $y \mapsto dy$, $x \mapsto 0$, $z \mapsto dy$

$du = x$

diagram commutes vertical maps $d^2 = 0$ horizontal maps are exact. δ_k is defined as follows.

Let $\underline{x} \in F''_k$ with $dx = 0$

<u>choose</u> y $\beta y = x$ then since $\beta(dy) = 0$

~~we may choose~~ $\exists!$ z $\alpha z = dy$ then $dz = 0$ since

$\alpha dz = d^2 y = 0$ and α is injective.

Let $\delta_k([x]) = [z] \in H_{k-1}(F'_\bullet)$

we made various choices, we need to show δ_k is independent of these choices.

- x' with $x - x' = du$ ~~z' with $\alpha z' = dy$~~
- y' with $\beta y' = x$

Proposition: If $0 \to F'_\bullet \xrightarrow{\alpha_\bullet} F_\bullet \xrightarrow{\beta_\bullet} F''_\bullet \to 0$ is a short exact sequence of complexes, then the following sequence is long exact:

$$\cdots \to H_{i+1}(F'_\bullet) \xrightarrow{\alpha_{i+1}} H_{i+1}(F_\bullet) \xrightarrow{\beta_{i+1}} H_{i+1}(F''_\bullet) \xrightarrow{\delta_{i+1}} H_i(F'_\bullet) \xrightarrow{\alpha_i} H_i(F_\bullet) \to \cdots$$

Do the diagram chase for each one.

"Homework / Midterm" will require reproducing some set of diagram chases.

Exercise / Lemmas:

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \\ 0 & \to & A' & \to & B' & \to & C' & \to & 0 \end{array}$$

commutative diagram with exact rows prove

SNAKE LEMMA:

$$0 \to \ker\alpha \to \ker\beta \to \ker\gamma \to \operatorname{coker}\alpha \to \operatorname{coker}\beta \to \operatorname{coker}\gamma \to 0$$

Five Lemma: If the following diagram commute and has exact rows

$$\begin{array}{ccccccccc} A_1 & \to & A_2 & \to & A_3 & \to & A_4 & \to & A_5 \\ \alpha_1\downarrow & & \beta_1\downarrow & & \gamma\downarrow & & \beta_2\downarrow & & \alpha_2\downarrow \\ B_1 & \to & B_2 & \to & B_3 & \to & B_4 & \to & B_5 \end{array}$$

β_i are $\cong$'s α_1 is surj. α_2 is inj then γ is $\cong$.

9-lemma:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A'' & \longrightarrow & B'' & \longrightarrow & C'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

commutative with exact columns and exact middle row

show if top or bottom row is exact, then the other is.

Derived functors.

In a category with kernels and cokernels (like R-modules) short exact sequences capture the notions

$$0 \to A \xrightarrow{\theta} B \xrightarrow{\phi} C \to 0$$

$A = \ker\phi$ is a subobject of B

$C = \operatorname{coker}\theta$ is the quotient of B by A.

Given a functor from $R\text{-mod} \xrightarrow{F} R\text{-mod}$ (or more generally between any 2 Abelian categories).

to what extent does F preserve short exact sequences? How can we measure the failure to preserve ~~ses~~ such?

Example Let $R = \mathbb{Z}$ so that $\{R\text{-mod}\} = \{\text{Abelian gps}\}$.

$M \otimes_R (-) : R\text{-modules} \longrightarrow R\text{-modules}$

so for example $R = \mathbb{Z}$ $M = \mathbb{Z}/4$

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$$

$$\Big\downarrow \ \mathbb{Z}/4 \otimes (-)$$

$$0 \to \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$$

fails exactness here

More generally we have

Lemma If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence of R-mods and M is an R-mod. Then

$$M \otimes_R A \xrightarrow{I_M \otimes f} M \otimes_R B \xrightarrow{I_M \otimes g} M \otimes_R C \to 0$$

is exact.

Def'n A Functor $F : R\text{mod} \to R\text{mod}$ is right-exact if $\forall$ s.e.s. $0 \to A \to B \to C \to 0$, $FA \to FB \to FC \to 0$ is exact.

pf. of Lemma. $I_M \otimes g$ is surjective.

let $\sum_i m_i \otimes c_i \in M \otimes C$ let $b_i \in B$ be s.t. $g(b_i) = c_i$

then $\sum_i m_i \otimes b_i \longmapsto \sum m_i \otimes c_i$

since $(Id_m \otimes g) \circ (Id_m \otimes f) = Id_m \otimes (g \circ f) = 0$

to prove $\mathrm{Im}(Id_m \otimes f) = \mathrm{Ker}(Id_m \otimes g)$ we need to show

$$M \otimes B \big/ \mathrm{Im}(Id_m \otimes f) \cong M \underset{R}{\otimes} C$$

Let $\sum_i m_i \otimes c_i \in M \otimes C$ choose $b_i \in B$ with $g(b_i) = c_i$

claim: the map $M \otimes C \longrightarrow M \otimes B \big/ \mathrm{Im}(Id_m \otimes f)$

$$\sum_i m_i \otimes c_i \longmapsto \sum_i m_i \otimes b_i$$

is well defined. Let b_i' be different choices. Then

$$\sum_i m_i \otimes b_i - \sum_i m_i \otimes b_i' = \sum_i m_i \otimes (b_i - b_i')$$

is in the image of $Id_m \otimes f$, since $\exists\, a_i$ with $f(a_i) = b_i - b_i'$

$\square$.

Given a right-exact functor F, we want a new functor $L_1 F$ (1st left derived functor) measuring failure of exactness on the left:

$$0 \to A \to B \to C \to 0$$

$$(L_1 F) C \xrightarrow{\delta} FA \to FB \to FC \to 0$$

Def'n S'pose F is a right exact functor on the category of R-modules. If A is an R-module, let

$$P_\bullet = \cdots \to P_i \xrightarrow{\phi_i} P_{i-1} \to \cdots \to P_0$$

be a projective resolution of A (e.g. $P_\bullet$ could be a free resolution). Then the ith ~~de~~ left derived functor of F applied to A is $(L_iF)A = H_i(FP_\bullet)$. i.e. the ith homology of the complex

$$FP_\bullet : \cdots \to FP_i \xrightarrow{F\phi_i} FP_{i-1} \to \cdots \to FP_0$$

Proposition The left derived functors are well defined (independent of choice of $P_\bullet$) and satisfy:

(a) $L_0F = F$

(b) If A is projective, then $(L_iF)A = 0 \ \forall i > 0$

(c) If $0 \to A \to B \to C \to 0$ is s.e.s. then there is a long exact sequence

$$\cdots \to L_{i+1}FA \to L_{i+1}FB \to L_{i+1}FC \xrightarrow{\delta_{i+1}} L_iFA \to L_iFB \to L_iFC \xrightarrow{\delta_i} \cdots \to L_1FC \xrightarrow{\delta_1} A \to B \to C \to 0$$

e.g. For $F = (M \otimes_R -)$

$L_i F$ is called $\mathrm{Tor}_i(M, -)$

let's compute tor_i in an example.

Let $x \in R$ be an element which is not a zero divisor compute $\mathrm{tor}_i(M, R/(x))$

$$0 \to R \xrightarrow{\cdot x} R \to R/(x) \to 0$$

so $\mathrm{Tor}_i(M, R/(x))$ is the homology of

$$0 \to R \otimes M \xrightarrow{\cdot x} R \otimes M \to 0$$
$$\quad \parallel$$
$$0 \to M \xrightarrow{\cdot x} M \to 0$$

note $\mathrm{Tor}_0(M, R/(x)) = M/xM \cong M \otimes_R R/(x)$

$\mathrm{Tor}_1(M, R/(x)) = \ker\{M \xrightarrow{\cdot x} M\}$

$= \{ m \in M \quad xm = 0 \}$

"x-torsion of M". e.g. if $R = \mathbb{Z}$ $x = N$ M abel. gp

$\mathrm{Tor}_1(M, \mathbb{Z}/N\mathbb{Z}) = N$-torsion in M.

Induced diagrams commute

(d) connecting homomorphism δ are "natural"

i.e. if
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

is a map of s.e.s. then the corresponding diagram of long exact seq. commutes;

$$\begin{tikzcd} \cdots \arrow[r] & L_iFA \arrow[r] \arrow[d] & L_iFB \arrow[r] \arrow[d] & L_iFC \arrow[r, "\delta_i"] \arrow[d] & L_{i-1}FA \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & L_iFA' \arrow[r] & L_iFB' \arrow[r] & L_iFC' \arrow[r] & L_{i-1}FA' \arrow[r] & \cdots \end{tikzcd}$$

pf: Recall that if $P_\bullet \to A$ $P'_\bullet \to A'$ are proj. resolutions and $\alpha : A \to A'$ is a module map, then $\exists$ a map of cxs $\alpha_\bullet : P_\bullet \to P'_\bullet$ inducing α, i.e. $H_i(\alpha_\bullet) : H_i(P_\bullet) \to H_i(P'_\bullet)$ is 0 for $i>0$ and α for $i=0$. Moreover $\alpha_\bullet$ is unique upto chain homotopy: if $\beta_\bullet : P_\bullet \to P'_\bullet$ also induces α, then $\alpha_\bullet - \beta_\bullet \simeq 0$

$$\begin{tikzcd} P_{i+1} \arrow[r, "d_{i+1}"] \arrow[d] & P_i \arrow[r, "d_i"] \arrow[d, "\alpha_i-\beta_i"] \arrow[dl, "h_i"'] & P_{i-1} \arrow[d] \arrow[dl, "h_{i-1}"'] \\ P'_{i+1} \arrow[r, "d'_{i+1}"'] & P'_i \arrow[r, "d'_i"'] & P'_{i-1} \end{tikzcd}$$

$$\alpha_i - \beta_i = h_{i-1} d_i + d'_{i+1} h_i$$
$$\Rightarrow H_i(\alpha_\bullet), H_i(\beta_\bullet) : H_i(P_\bullet) \to H_i(P'_\bullet)$$
same map.

Now suppose $A = A'$ so $P_\bullet$, $P'_\bullet$ are proj. res.s of the same A.

then $\exists$ $\alpha_0 : P_0 \to P_0'$ $\beta_0 : P_0' \to P_0$

inducing identity on A $\Rightarrow$ $\alpha_0 \circ \beta_0 : P_0' \to P_0'$

and $\beta_0 \circ \alpha_0 : P_0 \to P_0$ induce identity and

$\beta_0 \circ \alpha_0 - Id_{P_0} \simeq 0$ and $\alpha_0 \circ \beta_0 - Id_{P_0'} \simeq 0$

Since F is a functor $F\alpha_i : FP_i \to FP_i'$

is a map of complexes and thus induces maps

$H_i(F\alpha_0) : H_i(FP_0) \longrightarrow H_i(FP_0')$

it is an isomorphism since $H_i(F\beta_0)$ is its inverse.

Moreover, this isomorphism is canonical since if

$\tilde{\alpha}_0 : P_0 \longrightarrow P_0'$ is a different map of cx inducing id on A

then $\tilde{\alpha}_i - \alpha_i = hd + d'h$ so

$$F\tilde{\alpha}_i - F\alpha_i = (Fh)(Fd) + (Fd')(Fh)$$

so $H_i(F\tilde{\alpha}_0) = H_i(F\alpha_0) : H_i(FP_0) \longrightarrow H_i(FP_0')$

we prove (a) - (d) of proposition.

pf of (a) $P_1 \to P_0 \to A \to 0$ exact

since F is right exact we get

$$FP_1 \to FP_0 \to FA \to 0 \quad \text{exact}$$

$L_0FA = H_0(\cdots \to FP_1 \to FP_0 \to 0) = \text{coker}(FP_1 \to FP_0) = FA$

pf. of (b) If A is proj. then $0 \to A \to A \to 0$ is a resolution

so $(L_iF)A = H_i(0 \to 0 \to \dots \to 0 \to FA \to 0)$

pf of (c) s'pose $0 \to A' \xrightarrow{\alpha'} A \xrightarrow{\alpha} A'' \to 0$ is exact. We first constr. proj. resolutions $P'_\bullet \to A'$, $P_\bullet \to A$, $P''_\bullet \to A''$ and maps of complexes $\alpha'_\bullet : P'_\bullet \to P_\bullet$, $\alpha_\bullet : P_\bullet \to P''_\bullet$ which (1) induce α' and α and (2) the sequences $0 \to P'_i \xrightarrow{\alpha'_i} P_i \xrightarrow{\alpha_i} P''_i \to 0$ are short exact.

Let $P'_\bullet$ & $P''_\bullet$ be arbitrary and let $P_\bullet = P'_\bullet \oplus P'_\bullet$

$$\begin{tikzcd}
0 \arrow[r] & P'_1 \arrow[r] \arrow[d, "d'_1"'] & P'_1 \oplus P''_1 \arrow[r] \arrow[d, dotted] & P''_1 \arrow[r] \arrow[d, "d''_1"] & 0 \\
0 \arrow[r] & P'_0 \arrow[r] \arrow[d, "d'_0"'] & P'_0 \oplus P''_0 \arrow[r] \arrow[d] & P''_0 \arrow[r] \arrow[d, "d''_0"] \arrow[dll, dashed, "\gamma"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "\alpha'"'] & A \arrow[r, "\alpha"'] & A'' \arrow[r] & 0
\end{tikzcd}$$

we construct the differential on $P_\bullet = P'_\bullet \oplus P''_\bullet$ inductively

the 1st middle vertical map is $(\alpha' \circ d'_0) \oplus \gamma = d_0$

we need to show it is surjective. Let $a \in A$

then $\exists\ p'' \in P''_0$ s.t. $\alpha(a) = d''_0(p'')$

then $-\gamma(p'') + a \in \operatorname{Ker} \alpha$ since

$$\alpha(\gamma(-p'') + a) = -d''_0(p'') + \alpha(a) = 0$$

so $\exists\ a' \in A'$ with $\alpha'(a') = -\gamma(p'') + a$ then

let p'_0 be such that $d'_0(p'_0) = a'$ then $(p'_0, p'') \mapsto \gamma(-p'') + a + \gamma(p'') = a$

now proceed inductively, replacing the bottom row with the kernels of the newly constructed maps.

Recall that previously we showed that a short exact sequence of complexes ~~$0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$~~ gives us a long exact sequence in cohomology. Since $P_\bullet = P'_\bullet \oplus P''_\bullet$ the s.e.s. of cxs $0 \to P'_\bullet \to P'_\bullet \oplus P''_\bullet \to P''_\bullet \to 0$ gives us a s.e.s. of cxs $0 \to LP'_\bullet \to LP'_\bullet \oplus LP''_\bullet \to LP'_\bullet \to 0$

Consider the functor $\mathrm{Hom}_R(M, -) : R\text{mod} \to R\text{mod}$.

what does it do to s.e.s.'s? $R = \mathbb{Z}$

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$$

$$\mathrm{Hom}(\mathbb{Z}/2, -) \quad 0 \to 0 \xrightarrow{\cdot 0} 0 \to \mathbb{Z}/2 \to 0$$

technically exact here   definitely not exact here

$$0 \to \mathbb{Z}/2 \xrightarrow{\cdot 2} \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0 \qquad 1 \mapsto 1$$

$$\begin{array}{ccccccc}
0 \to & \mathrm{Hom}(\mathbb{Z}/2, \mathbb{Z}/2) & \to & \mathrm{Hom}(\mathbb{Z}/2, \mathbb{Z}/4) & \to & \mathrm{Hom}(\mathbb{Z}/2, \mathbb{Z}/2) & \\
& \wr\| & & \wr\| & & & \\
0 \to & \mathbb{Z}/2 & \xrightarrow{\text{id}} & \mathbb{Z}/2 & \xrightarrow{0} & \mathbb{Z}/2 & \to 0
\end{array}$$

$(1 \mapsto 1) \longmapsto (1 \mapsto 2)$

$(1 \mapsto 2) \longmapsto (1 \mapsto 2) = 0$

Lemma $\mathrm{Hom}_R(M,-)$ is left exact, i.e. $\forall$ s.e.s.

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

$$0 \to \mathrm{Hom}(M,A) \xrightarrow{i_*} \mathrm{Hom}(M,B) \xrightarrow{j_*} \mathrm{Hom}(M,C) \quad \text{is exact}$$

pf.

$$\begin{tikzcd} & & M \arrow[dll, dotted, "\exists!"'] \arrow[d, "f"] \arrow[dr, "0"] & & \\ 0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r, "j"] & C \arrow[r] & 0 \end{tikzcd}$$

$\in \ker i_* \Rightarrow \forall m \in M \quad i\phi(m) = 0 \Rightarrow \phi(m) = 0 \Rightarrow \phi = 0.$

suppose $f \in \ker j_* \quad j \circ f = 0 \Rightarrow \forall m \in M \quad j(f(m)) = 0$

$\Rightarrow \exists!\ a \in A$ s.t. $i(a) = f(m)$ so let $g \in \mathrm{Hom}(M,A)$

be $g(m) = a$ then $f = i \circ g$. $\blacksquare$.

When is $\mathrm{Hom}(M,-)$ exact on the right as well?

given $h \in \mathrm{Hom}(M,C)$ we want f with $h = j \circ f$

$$\begin{tikzcd} & M \arrow[dl, dotted, "\exists f"'] \arrow[d, "h"] & \\ B \arrow[r, two heads] & C \arrow[r] & 0 \end{tikzcd} \quad \Longleftrightarrow \quad M \text{ is projective.}$$

$\mathrm{Hom}_R(M,-)$ is left exact and fully exact iff M is proj.

The functor $\mathrm{Hom}_R(-,M)$ is <u>contravariant</u> it reverses the direction of arrows

Lemma $\mathrm{Hom}_R(-, M)$ is left exact, i.e. $\forall$

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$$

$$0 \to \mathrm{Hom}(C, M) \xrightarrow{j^*} \mathrm{Hom}(B, M) \xrightarrow{i^*} \mathrm{Hom}(A, M) \text{ is exact.}$$

pf.

$$\begin{tikzcd}
0 & C \arrow[l] \arrow[dr, "f"'] & B \arrow[l, "j"'] \arrow[d, "g"] & A \arrow[l, "i"'] \arrow[dl, "0"] & 0 \arrow[l] \\
 & & M & &
\end{tikzcd}$$

- s'pose $f: C \to M$ has $j^* f = 0 \Rightarrow \forall b \in B \;\; f(j(b)) = 0$

  $\Rightarrow f(c) = 0 \;\; \forall c \;\; \Rightarrow f = 0$

- s'pose $i^* g = 0 \Rightarrow \forall a \;\; g(i(a)) = 0$ we define $f: C \to M$

as follows, for $c \in C$ choose $b \in B$ $j(b) = c$ let

$f(c) = g(b)$ well-defined: s'pose $b' \in B$ $j(b') = c$

$$f(c) = g(b') = g(b) + g(b' - b) = g(b) + g(i(a)) = g(b).$$

For $\mathrm{Hom}(-, M)$ to be fully exact we need

$\forall \;\; h: A \to M \;\; \exists \;\; g: B \to M$ with $h = g \circ i$

$$\begin{tikzcd}
B & A \arrow[l, hook', "i"'] \arrow[d, "h"] \\
 & M \arrow[from=1-1, dotted, "h"']
\end{tikzcd}$$

$\Longleftrightarrow$ M is an injective module

In order to define right derived functors (of left exact functors) we need injective resolutions

$$\begin{tikzcd} & P \arrow[d] \arrow[dl, dotted] & \\ B \arrow[r, two heads] & A \arrow[r] & 0 \end{tikzcd}
\qquad
\begin{tikzcd} & I \quad f(a)/k & \\ B \arrow[ur, dotted] & A \arrow[l, hook'] \arrow[u, "f"'] & 0 \arrow[l] \end{tikzcd}$$

$b = ka$

P is projective iff this diagram always completes

I is injective iff this diagram always completes.

Examples of Projective modules arise in nature: free modules.

Injective modules are less familiar

e.g. if $R = \mathbb{Z}$ $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$ are examples of injective modules (not finitely generated as Abelian gps). need to be able to divide.

<u>Def'n</u> an <u>injective resolution</u> of an R-mod M is a sequence $M \to I_0 \to I_1 \to I_2$ s.t. I_k is inj.

and $0 \to M \to I_0 \to I_1 \cdots$ is exact.

$1 \mapsto a$

e.g. $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$

$1 \mapsto 1/a$

or

$$0 \to \mathbb{Z}/N \to \mathbb{Q}/\mathbb{Z} \xrightarrow{\cdot N} \mathbb{Q}/\mathbb{Z} \to 0$$

$1 \longmapsto 1/N$

Prop. (we won't prove) R-mod has enough injectives (inj. res. always exist).

Meta mathematics. All the work we did for projective resolutions and left derived functors works exactly the same for injective resolutions and right derived functors just with all the arrows reversed!

Can be formalized using opposite categories: e.g. in $R\text{-mod}^{op}$ proj. resolution become injective resolutions.

Let $F : R\text{mod} \to R\text{mod}$ be a left exact covariant functor.

Let A be an R-mod with an injective resolution

$$0 \to A \to I_0 \to I_{-1} \to \cdots$$

Def'n The ith right derived functor R^iF is given by

$$(R^iF)(A) = H_{-i}(FI_\bullet)$$

Theorem R^iF is well defined (indep of $I_\bullet$)

(a) $R^0F = F$ (b) if A is injective then $R^kFA = 0$ $k > 0$

(c) If $0 \to A' \to A \to A'' \to 0$ s.e.s. then

$$0 \to FA' \longrightarrow FA \longrightarrow FA'' \to R^1FA' \to R^1FA \to R^1FA'' \to R^2FA' \to \cdots$$

is long exact.

e.g. $\mathrm{Ext}^i(A,B) = R^i \mathrm{Hom}(A,-)$

compute $\mathrm{Ext}^k(\mathbb{Z}/n, \mathbb{Z}/m)$

$$0 \to \mathbb{Z}/m \longrightarrow \mathbb{Q}/\mathbb{Z} \xrightarrow{\cdot m} \mathbb{Q}/\mathbb{Z} \to 0 \qquad \text{inj. res.}$$
$$1 \longmapsto 1/m$$

so $\mathrm{Ext}^k(\mathbb{Z}/n, \mathbb{Z}/m)$ computed by complex

$$\mathrm{Hom}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\cdot m} \mathrm{Hom}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z})$$
$$(1 \mapsto a/n) \longmapsto (1 \mapsto ma/n)$$
$$a \in \mathbb{Z}/n \qquad\qquad ma \in \mathbb{Z}/n$$

$$0 \to \mathbb{Z}/n \xrightarrow{\cdot m} \mathbb{Z}/n \to 0$$

$$\mathrm{Ext}^0(\mathbb{Z}/n, \mathbb{Z}/m) = \ker(\mathbb{Z}/n \xrightarrow{\cdot m} \mathbb{Z}/n) \overset{(*)}{\cong} \mathbb{Z}/d$$
$$\mathrm{Ext}^1(\mathbb{Z}/n, \mathbb{Z}/m) = \mathbb{Z}/(n,m) = \mathbb{Z}/d \qquad d = \gcd(n,m)$$

(*) $\mathbb{Z}/d \longrightarrow \{k \in \mathbb{Z}/n : mk = 0\}$
$$1 \longmapsto n/d$$

If F is contravariant left exact, i.e.

$$0 \to A \to B \to C \to 0 \quad \text{s.e.s.}$$

$\Rightarrow$ $0 \to FC \to FB \to FA$ exact

Then the right derived functors are defined with <u>projective resolutions</u>. Let

$$\cdots\cdots \to P_1 \to P_0 \to A \quad \text{be a proj res.}$$

$$(R^k F)(A) = H^k(FP_\bullet) := H_{-k}(FP_{-\bullet})$$

$$0 \to FP_0 \to FP_1 \to FP_2 \to \cdots$$

$$\text{deg } 0 \qquad \text{deg } -1 \qquad \text{deg } -2$$

$\mathrm{Hom}(-, B)$ is an example of a contrav. left exact functor.

we will later show

~~Ex~~ $R^k \mathrm{Hom}(-, B) = \mathrm{Ext}^k(-, B)$

so $\mathrm{Ext}^k(A, B)$ can be computed with either a proj resolution of A or an inj. resolution of B.

Why is it called ~~Ext~~?

<u>Jargon</u> If X can be written in a s.e.s.

$$\alpha: \quad 0 \to B \to X \to A \to 0$$

we say "X is given as an extension of A by B".

$X = A \oplus B$ (with the obvious maps) is called the <u>trivial extension</u>

two extensions α, α' are <u>equivalent</u> if $\exists$ a commutative diagram:

$$\begin{tikzcd}
\alpha: & 0 \arrow[r] & B \arrow[r] \arrow[d, "id_B"'] & X \arrow[r] \arrow[d, "f"] & A \arrow[r] \arrow[d, "id_A"] & 0 \\
\alpha': & 0 \arrow[r] & B \arrow[r] & X' \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

$f: X \to X'$ is an isomorphism. Note we can have

$X \cong X'$ but $\alpha \not\equiv \alpha'$.

Let $E^1_R(A,B)$ be the set of equivalence classes of extensions

We will show $\mathrm{Ext}^1_R(A,B) = E^1_R(A,B)$ (as sets)

moreover, we can construct module structure on $E^1_R(A,B)$

so $=$ is as R-mods, moreover can construct $E^1_R(-,B)$

and $E^1_R(A,-)$ as functors.

<u>Lemma</u> $E^1_R(-,B)$: R-mod $\to$ Sets is a contravariant functor.

<u>Pf.</u> given $v: A' \to A$ and an extension

$\alpha: \ 0 \to B \to X \xrightarrow{f} A \to 0$ we need to

construct $\alpha' = v^*\alpha$ (more streamlined notation for

$$E^1_R(v,B): E^1_R(A,B) \to E^1_R(A',B)$$
$$\alpha \mapsto \alpha'$$

$\alpha': \ 0 \to B \to X' \to A' \to 0$

we let X' be the fiber product or Cartesian product

of $X \xrightarrow{f} A$ and $A' \xrightarrow{v} A$:

$u : B \to B'$

functoriality in B $\quad E'(A,B) \xrightarrow{u_*} E'(A,B')$

$$\begin{tikzcd}
\alpha & 0 \arrow[r] & B \arrow[r, "g"] \arrow[d, "u"'] & X \arrow[r] \arrow[d] \arrow[dl, phantom, "\lrcorner"] & A \arrow[d, equal] \\
u_*\alpha & 0 \arrow[r] & B' \arrow[r] & X' \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

given
$$\begin{tikzcd}
& A \arrow[d, "g"] \\
B \arrow[r, "f"'] & Y
\end{tikzcd}$$
pullback is
$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & A \arrow[d, "g"] \\
B \arrow[r] & Y
\end{tikzcd}$$

where $0 \to X \to A \oplus B \to Y$

given
$$\begin{tikzcd}
X \arrow[r] \arrow[d] & A \\
B
\end{tikzcd}$$
pushout is
$$\begin{tikzcd}
X \arrow[r] \arrow[d] & A \arrow[d] \\
B \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & Y
\end{tikzcd}$$

where $X \to A \oplus B \to Y \to 0$

---

satisfy universal properties

$$X' = \{ (x, a') \in X \oplus A' : f(x) = v(a') \}$$

$$\begin{tikzcd} & & X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & A' \arrow[r] \arrow[d, "v"] & 0 \\ 0 \arrow[r] & B \arrow[r] & X \arrow[r, "f"] & A \arrow[r] & 0 \end{tikzcd}$$

by construction $\operatorname{Ker}(X' \to A') = \{ (x, a') : f(x) = v(a') \ \ a' = 0 \}$

$$= \{ x : f(x) = 0 \} = B$$

so we get an extension

$$\alpha' : 0 \to B \to X' \to A' \to 0$$

This allows us to define $\varepsilon : \operatorname{Ext}^1(A, B) \to E^1(A, B)$

Let $B \to I_\bullet$ be an injective resolution

$[\phi] \in \operatorname{Ext}^1(A, B)$ is represented by

$$\phi \in \operatorname{Ker}\left(\operatorname{Hom}(A, I_{-1}) \to \operatorname{Hom}(A, I_{-2}) \right)$$

$$\begin{tikzcd} & & & A \arrow[dl, dotted, "\tilde{\phi}"'] \arrow[d, "\phi"] \arrow[dr, "0"] & \\ 0 \arrow[r] & B \arrow[r] & I_0 \arrow[r] & I_{-1} \arrow[r] & I_{-2} \end{tikzcd}$$

The lift $\tilde{\phi}$ is well defined on $\tilde{\phi} : A \to I_0/B$

define $\varepsilon([\phi]) = \tilde{\phi}^*(\alpha)$ where

$$\alpha : \quad 0 \to B \to I_0 \to I_0/B \to 0 \qquad \alpha \in E^1(I_0/B, B)$$

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & A \arrow[r] \arrow[d, "\tilde{\phi}"] & 0 \\ 0 \arrow[r] & B \arrow[r] & I_0 \arrow[r] & I_0/B \arrow[r] & 0 \end{tikzcd}$$

lots to check!
ε well defined,
bijective. ε is
natural. (ε is an equivalence
of functors).

The inverse to ε: given

$$\begin{tikzcd}
\alpha: 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, dashed, "①"] & A \arrow[r] \arrow[d, dashed, "② \tilde{\phi}"] & 0 \\
0 \arrow[r] & B \arrow[r] & I_0 \arrow[r] & I_0/B \arrow[r] & 0
\end{tikzcd}$$

① exists because I_0 is injective

② exists by a diagram chase

then let ϕ be the composition

$$\begin{tikzcd}
& & A \arrow[dl, "\tilde{\phi}"'] \arrow[d, "\phi"] & \\
0 \arrow[r] & I_0/B \arrow[r] & I_{-1} \arrow[r] & I_{-2}
\end{tikzcd}$$

$$\varepsilon^{-1}(\alpha) = [\phi] \in \ker\Big(\mathrm{Hom}(A, I_{-1}) \to \mathrm{Hom}(A, I_{-2})\Big)$$

Module structure on $E_R^1(A,B)$:

we need to define $r[\alpha]$ and $[\alpha]+[\alpha']$ for

$r \in R$ $\alpha, \alpha' \in E_R^1(A,B)$

since r defines maps $A \xrightarrow{\cdot r} A$ and

$B \xrightarrow{\cdot r} B$ functorality gives us maps

$r^*[\alpha]$ and $r_*[\alpha]$ either of which define $r.[\alpha]$.

to define $[\alpha]+[\alpha']$ consider

$$\alpha \oplus \alpha' : 0 \to B\oplus B \to X \oplus X' \to A \oplus A \to 0$$

$$\Delta: A \to A\oplus A \qquad \sigma: B\oplus B \to B$$
$$a \mapsto (a,a) \qquad (b_1, b_2) \mapsto b_1 + b_2$$

Since $[\alpha \oplus \alpha'] \in E^1_R(A \oplus A, B \oplus B)$

we define Baer sum: $[\alpha] + [\alpha'] = \Delta^* \sigma_* (\alpha \oplus \alpha')$

$$E^1_R(A\oplus A, B\oplus B) \xrightarrow{\sigma_*} E^1_R(A\oplus A, B) \xrightarrow{\Delta^*} E^1_R(A, B)$$

$$\begin{tikzcd}
0 \arrow[r] & B\oplus B \arrow[r] \arrow[d, "\sigma"] \arrow[dr, phantom, "\lrcorner"] & X\oplus X' \arrow[r] \arrow[d] & A\oplus A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] & X'' \arrow[r] & A\oplus A \arrow[r] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[u, equal] & X''' \arrow[r] \arrow[u] \arrow[ur, phantom, "\square"] & A \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Lot's of work, but straightforward $E^1_R(-,-) : R\text{mod} \times R\text{mod} \to R\text{mod}$

and $\mathrm{Ext}^1_R(-,-) : R\text{mod} \times R\text{mod} \to R\text{mod}$ equivalent bifunctors.
(derived category makes this much easier!)

Higher Yoneda Exts:

We can define $E^k_R(A,B)$ to be the set of equivalence classes

$$\begin{tikzcd}
\alpha: & 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_k \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
\alpha': & 0 \arrow[r] & B \arrow[r] & X_1' \arrow[r] & X_2' \arrow[r] & \cdots \arrow[r] & X_k' \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

any commutative diagram.

$\alpha \sim \alpha'$ is the equivalence relation generated by the above

Turns out $E^k_R(A,B) \cong \mathrm{Ext}^k_R(A,B)$

From this point of view, there is an easily defined multiplication (Yoneda product)

$$\mathrm{Ext}^n_R(B,C) \otimes_R \mathrm{Ext}^m_R(A,B) \to \mathrm{Ext}^{n+m}_R(A,C)$$

$$\alpha \otimes \beta \longmapsto \alpha\beta$$

α $\quad 0 \to C \to X_1 \to \cdots \to X_n \to B \to 0$

β $\quad 0 \to B \to Y_1 \to \cdots \to Y_m \to A \to 0$

$$\alpha\beta \quad \begin{tikzcd}[column sep=small] & & & & & B \arrow[dr] & & & & & & \\ 0 \arrow[r] & C \arrow[r] & X_1 \arrow[r] & \cdots \arrow[r] & X_n \arrow[ur] \arrow[rr] & & Y_1 \arrow[r] & Y_2 \arrow[r] & \cdots \arrow[r] & Y_m \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

product is associative but not commutative

$$\mathrm{Ext}^*_R(A,A) = \bigoplus_{n \geq 0} \mathrm{Ext}^n_R(A,A)$$ non-commutative graded ring (difficult to compute in general).

We are often in a situation where we have a map of complexes $\alpha_\bullet : F_\bullet \to G_\bullet$ and we would like to understand the induced maps on homology

$$H_i(F_\bullet) \longrightarrow H_i(G_\bullet)$$

if $\alpha_\bullet$ were part of a s.e.s. of cxs (i.e. all α_i's were injective or all α_i's were surjective) then we would get the long exact sequence in homology.

There is a useful construction which gives us a s.e.s. of complexes making the above maps the connecting homomorphisms

<u>Def'n / Thm</u> Let $\alpha_\bullet : F_\bullet \to G_\bullet$ be a map of cx's.

The <u>mapping cone</u> of α is the complex $M_\bullet(\alpha_\bullet)$

$$M_k(\alpha) = G_k \oplus F_{k-1} \xrightarrow{d_k^m} G_{k-1} \oplus F_{k-2}$$

$$(a, b) \longmapsto (d_k^G a + \alpha_{k-1} b, -d_{k-1}^F b)$$

Then we get a s.e.s. of complexes

$$0 \to G_\bullet \to M_\bullet \to F_\bullet[-1] \to 0$$

such that the connecting homomorphisms for the long exact sequence are given by $H(\alpha)$.

$$F_j[i] = F_{i+j} \text{ with diff } (-1)^i d$$

$$\begin{tikzcd}
\delta : H_k(F_\bullet[-1]) \arrow[r] \arrow[d, equal] & H_{k-1}(G_\bullet) \\
H_{k-1}(F_\bullet) \arrow[ur, "\cancel{H_{k-1}}\ H_{k-1}(\alpha)"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & G_{k+1} \arrow[r] \arrow[d, "d"] & G_{k+1} \oplus F_k \arrow[r] \arrow[d, "\begin{pmatrix} d & \alpha \\ 0 & -d \end{pmatrix}"] & F_k \arrow[r] \arrow[d, "-d"] & 0 \\
0 \arrow[r] & G_k \arrow[r] \arrow[d] & G_k \oplus F_{k-1} \arrow[r] \arrow[d, "\begin{pmatrix} d & \alpha \\ 0 & -d \end{pmatrix}"] & F_{k-1} \arrow[r] \arrow[d, "-d"] & 0 \\
0 \arrow[r] & G_{k-1} \arrow[r] & G_{k-1} \oplus F_{k-2} \arrow[r] & F_{k-2} \arrow[r] & 0
\end{tikzcd}$$

x

$(0, x) \downarrow (\alpha(x), dx)$

Introduction to the derived category:

The dualization functor $F = \mathrm{Hom}(-, \mathbb{Z})$ on $\mathbb{Z}$-modules was an example of a left exact functor.

To define the derived functors, we applied it to a proj. resolution, then took (co)homology:

Module	projective cx	(co)homology
$\mathbb{Z}/2$	$0 \to \underset{1}{\mathbb{Z}} \xrightarrow{\cdot 2} \underset{0}{\mathbb{Z}} \to 0$	$H_1 = 0 \quad H_0 = \mathbb{Z}/2$
$\Big\downarrow \mathrm{Hom}(-,\mathbb{Z})$	$\Big\downarrow \mathrm{Hom}(-,\mathbb{Z})$	
0	$0 \leftarrow \underset{1}{\mathbb{Z}} \xleftarrow{\cdot 2} \underset{0}{\mathbb{Z}} \leftarrow 0$	$H^1 = \mathbb{Z}/2 \quad H^0 = 0$

F applied to cx didn't loose info (e.g. applying F twice gets us back to where we started). Applying F directly to the (co)homology of cx did loose info.

Lesson: complexes good (co)homology bad

Idea of derived category: work directly with complexes instead of their homology. Replace modules with complexes of projective modules (or inj).

Since we are replacing modules with ~~resolutions~~ ~~we~~ complexes having the module as its homology, we want this to be functorial:

- maps of complexes inducing the same map in homology should be considered the same in our category
- maps inducing an isomorphism on homology should be isomorphisms in our category (in particular, invertible).

Category	Objects	Morphisms	Properties
$\mathcal{M}$	R-modules	R-mod homomorphisms	Abelian
$Cx(\mathcal{M})$	cx's of R-mods	maps of complexes quotient $\downarrow$	Abelian
$K(\mathcal{M})$	"	homotopy classes of maps of complexes localization $\downarrow$	Not Abelian Triangulated
$D(\mathcal{M})$	"	homotopy classes of maps and formal inverses of quasi-isomorphisms	Triangulated

<u>Def'n</u> The homotopy category of complexes $K(\mathcal{M})$ is the category whose objects are cxs and morphisms are homotopy classes of maps between cxs

$$\begin{tikzcd} \cdots \arrow[r] & A_i \arrow[r, "d"] \arrow[d, "\alpha_i"] \arrow[dl, "h"'] & A_{i-1} \arrow[r] \arrow[d, "\alpha_{i-1}"] \arrow[dl, "h"'] & \cdots \arrow[dl] \\ \cdots \arrow[r] & B_i \arrow[r] & B_{i-1} \arrow[r] & \cdots \end{tikzcd}$$

$$\alpha_\bullet \simeq \alpha'_\bullet \iff \exists\, h \text{ s.t. } \alpha_i - \alpha'_i = \overset{\deg 1}{h}\overset{\deg -1}{d} \pm dh$$

Note that if we pick a proj. resolution $P_\bullet(M)$ for each $M \in \mathcal{M}$ we get a well defined functor

$$P : \mathcal{M} \longrightarrow K(R\text{-mod})$$

(not well defined to $Cx(\mathcal{M})$!)

Note also that the homology functors $H_i : K(\mathcal{M}) \to \mathcal{M}$ are well defined since homotopic chain maps induce the same map on homology

However there is a cost to passing to the homotopy category

$CX(\mathcal{M})$ is Abelian, $K(\mathcal{M})$ is <u>not</u>

<u>example</u>: consider the following morphism of cx s (which are concentrated in degree 0)

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
\mathbb{Z}/4 \arrow[r] \arrow[d] & \mathbb{Z}/2 \arrow[d] \\
0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

~~[illegible]~~ If $K(m)$ we Abelian, this map would have a kernel, indeed we would get a monomorphism:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
\mathbb{Z}/2 \arrow[r, hook] \arrow[d] & \mathbb{Z}/4 \arrow[d] \\
0 \arrow[r] & 0
\end{tikzcd} \qquad (*)$$

(in $CX(\mathcal{M})$, this is a monomorphism).

In an Abelian category if $A \overset{\alpha}{\hookrightarrow} B$ is a monomorphism and it factors

$$\begin{tikzcd}
A \arrow[rr, hook, "\alpha"] \arrow[dr, "\gamma"'] & & B \\
& C \arrow[ur, "\delta"'] &
\end{tikzcd}$$

then both γ & δ are monomorphisms.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
\mathbb{Z}/2 \arrow[r, "id"] \arrow[d] & \mathbb{Z}/2 \arrow[r, hook] \arrow[d] & \mathbb{Z}/4 \arrow[d] \\
0 \arrow[r] \arrow[d] & \mathbb{Z}/4 \arrow[r] \arrow[d] \arrow[ur, dotted] & 0 \arrow[d] \\
0 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

factors $(*)$ but second map is homotopic to zero!

In an Abelian Cat if $A \hookrightarrow B$ is a monomorphism

and
$$\begin{tikzcd} A' \arrow[r, "\alpha"] \arrow[rr, bend right, "0"'] & A \arrow[r, hook] & B \end{tikzcd} \Rightarrow \alpha = 0$$

$$\begin{tikzcd}
\vdots & & \\
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
\mathbb{Z}/2 \arrow[r, "id"] \arrow[d] & \mathbb{Z}/2 \arrow[r, hook] \arrow[d] & \mathbb{Z}/4 \arrow[d] \\
\mathbb{Z}/4 \arrow[r] \arrow[ur, dotted, "?"] & 0 \arrow[r] & 0 \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

$$A' \xrightarrow{\alpha} A \longrightarrow B$$

Composition: is homotopic to zero

$$\begin{tikzcd}
A' \arrow[r] & B \\
0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
\mathbb{Z}/2 \arrow[r, hook] \arrow[d] & \mathbb{Z}/4 \arrow[d] \\
\mathbb{Z}/4 \arrow[r] \arrow[d] \arrow[ur, "id"] & 0 \arrow[d] \\
0 \arrow[r] & 0
\end{tikzcd}$$

but α is not homotopic to zero.

$K(m)$ doesn't have short exact sequences, instead

it has <u>exact triangles</u>

$$A_\bullet \longrightarrow B_\bullet \longrightarrow C_\bullet \longrightarrow A_\bullet[-1]$$

$$\begin{tikzcd} A \arrow[r] & B \arrow[dl] \\ C \arrow[u, "-1"] & \end{tikzcd}$$

they are given by the ~~m~~ mapping cones

$$A \xrightarrow{f} B \longrightarrow \mathrm{Cone}(f) \longrightarrow A[-1]$$

$$\begin{tikzcd}
A_{i+1} \arrow[r, "f_{i+1}"] \arrow[d] & B_{i+1} \arrow[r] \arrow[d] & B_{i+1} \oplus A_i \arrow[r] \arrow[d, "\begin{pmatrix} d & f_i \\ 0 & -d \end{pmatrix}"] & A_i \arrow[d] \\
A_i \arrow[r, "f_i"] & B_i \arrow[r] & B_i \oplus A_{i-1} \arrow[r] & A_{i-1}
\end{tikzcd}$$

exact triangles induce long exact sequences in (co)homology

$K(m)$ satisfies axioms of a <u>triangulated category</u>

Def'n A map of complexes $A_\bullet \xrightarrow{\alpha} B_\bullet$ is a quasi-isomorphism (qis) if $H_i(\alpha) : H_i(A_\bullet) \xrightarrow{\cong} H_i(B_\bullet)$

Def'n $D(\mathcal{M})$ is $K(\mathcal{M})$ localized at qis (general categorical construction where we formally add inverses to all qis's). In general, localized categories are unweildy ~~tok~~ because of the word problem but all morphisms in $D(\mathcal{M})$ are of the form

$f \circ s^{-1}$ f map of complexes, s a q-iso

$$\begin{tikzcd}
 & B_\bullet \arrow[dl, "s"'] \arrow[dr, "f"] & \\
A_\bullet \arrow[rr, "f \circ s^{-1}"'] & & C_\bullet
\end{tikzcd}$$

example s'pose $A_\bullet$ has $H_k(A_\bullet) = \begin{cases} M & k = 0 \\ 0 & k \neq 0 \end{cases}$

then $A_\bullet \cong [0 \to M \to 0]$ in $D(\mathcal{M})$

$$\begin{tikzcd}
\cdots \arrow[r] & A_{-1} \arrow[r, "d_{-1}"] & A_0 \arrow[r, "d_0"] & A_1 \arrow[r] & \cdots \\
\arrow[r] & A_{-1} \arrow[r] \arrow[u] \arrow[d] & \operatorname{Ker} d_0 \arrow[r] \arrow[u] \arrow[d] & 0 \arrow[r] \arrow[u] \arrow[d] & {} \\
\arrow[r] & 0 \arrow[r] & M \arrow[r] & 0 \arrow[r] & {}
\end{tikzcd}$$

q iso

q-iso

If $F : \mathcal{M} \to \mathcal{M}$ is a functor. ~~which~~ Then it extends to $F : Cx(\mathcal{M}) \to Cx(\mathcal{M})$

& $F : K(\mathcal{M}) \to K(\mathcal{M})$

if F is additive (sums go to sums) it preserves triangles and is a functor of triangulated categories.

However F may not extend to $D(\mathcal{M})$ because F may not preserve q-iso's. If F is exact it does, if F is only right exact we solve the problem as follows.

~~The problem is now~~ Let $\mathcal{P} \subset \mathcal{M}$ be the subcategory of projective modules. We then get $CX(\mathcal{P}), K(\mathcal{P}), D(\mathcal{P})$

Two key points

- $K(\mathcal{P}) \simeq D(\mathcal{P})$ every q-iso is a homotopy equiv. (generalizes the case of resolutions)
- There exists a "projective resolution functor" $D^+(\mathcal{M}) \xrightarrow[\simeq]{} K^+(\mathcal{P})$ which is an equivalence of categories. (generalizing the existance of a proj. resolution).

We then define

$$\begin{tikzcd} L.F : D^+(\mathcal{M}) \arrow[r] \arrow[d] & D^+(\mathcal{M}) \\ K^+(\mathcal{P}) \arrow[r, "F"] & K^+(\mathcal{P}) \arrow[u, hook] \end{tikzcd}$$

$L_\bullet F(A_\bullet)$ is a <u>complex</u> its (co)homology gives traditional derived functors.

example of how this is helpful. The left derived functor of $(-)\otimes(-)$ is denoted $\overset{L}{\otimes}$:

$$R/_{(x)} \overset{L}{\otimes} M = [R \xrightarrow{\cdot x} R] \otimes M = [M \xrightarrow{\cdot x} M] \in D(\mathfrak{m})$$

or if $[\cdots \xrightarrow{\alpha_2} P_1 \xrightarrow{\alpha_1} P_0 \to 0] \cong M$ then

$$R/_{(x)} \overset{L}{\otimes} M \cong [\cdots \to P_1/_{xP_1} \to P_0/_{xP_0} \to 0] \in D(\mathfrak{m})$$

claim: these are isomorphic in $D(\mathfrak{m})$. Why? They are both q-iso to $[R \xrightarrow{\cdot x} R] \otimes [\cdots \to P_1 \to P_0 \to 0]$

$$\begin{tikzcd}
 & 0 \arrow[r] & M \arrow[r, "\cdot x"] & M \arrow[r] & 0 & \text{q iso} \\
\cdots \arrow[r] & P_2 \oplus P_1 \arrow[u] \arrow[d] \arrow[r, "{\begin{pmatrix} \alpha_2 & 0 \\ \cdot x & -\alpha_1 \end{pmatrix}}"'] & P_1 \oplus P_0 \arrow[u] \arrow[d] \arrow[r, "{\begin{pmatrix} \alpha_1 \\ \cdot x \end{pmatrix}}"'] & P_0 \arrow[u] \arrow[d] \arrow[r] & 0 & \\
\cdots \arrow[r] & P_2/_{xP_2} \arrow[r] & P_1/_{xP_1} \arrow[r] & P_0/_{xP_0} \arrow[r] & 0 & \text{q iso}
\end{tikzcd}$$

Composition: $L(F \circ G) = LF \circ LG$ replaces spec. seqs.

example let $M^\vee := R^\bullet \mathrm{Hom}(M, R)$ derived dual

$$R^\bullet \mathrm{Hom}(A, B) = A^\vee \overset{L}{\otimes} B$$ (like vector spaces!)