Homological Algebra Homework problem set 0 Math 423

In the following problems, elements of direct sums will be considered as row vectors and the maps will be expressed as matrices, acting on row vectors by multiplication on the right.

In your answers, express all vector spaces and Abelian groups in standard form. Namely, all (finite dimensional) complex vector spaces are of the form $\mathbb{C}^d$ for some $d \geq 0$, and all (finitely generated) Abelian groups are products of cyclic groups, i.e. of the form $\mathbb{Z}^d \times (\mathbb{Z}/p_1)^{a_i} \times \cdots \times (\mathbb{Z}/p_k)^{a_k}$ for some $d \geq 0$, primes $p_1, \ldots, p_k$, and $a_1, \ldots, a_k > 0$.

1. Compute the homology groups of the following complex of $\mathbb{C}$-modules (a.k.a. complex vector spaces). The first non-trivial one is in degree 3, the last is in degree 0:

$$0 \longrightarrow \mathbb{C}^2 \xrightarrow{\left(\begin{smallmatrix} 9 & 9 \\ 6 & 6 \end{smallmatrix}\right)} \mathbb{C}^2 \xrightarrow{\left(\begin{smallmatrix} -5 & 10 \\ 5 & -10 \end{smallmatrix}\right)} \mathbb{C}^2 \xrightarrow{\left(\begin{smallmatrix} 10 & 6 \\ 5 & 3 \end{smallmatrix}\right)} \mathbb{C}^2 \longrightarrow 0$$

2. Compute the homology groups of the following complex of $\mathbb{Z}$-modules (a.k.a. Abelian groups). The first non-trivial one is in degree 3, the last is in degree 0:

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\left(\begin{smallmatrix} 9 & 9 \\ 6 & 6 \end{smallmatrix}\right)} \mathbb{Z}^2 \xrightarrow{\left(\begin{smallmatrix} -5 & 10 \\ 5 & -10 \end{smallmatrix}\right)} \mathbb{Z}^2 \xrightarrow{\left(\begin{smallmatrix} 10 & 6 \\ 5 & 3 \end{smallmatrix}\right)} \mathbb{Z}^2 \longrightarrow 0$$

3. Let R be the ring $\mathbb{C}[x, y]$. Compute the homology groups of the following complex of R modules (the first non-trivial module is in degree 2, the last non-trivial module is in degree 0):

$$0 \to R \xrightarrow{\alpha} R \oplus R \xrightarrow{\beta} R \to 0$$

where $\alpha : f(x, y) \mapsto (x \cdot f(x, y), y \cdot f(x, y))$ and $\beta : (g(x, y), h(x, y)) \mapsto y \cdot g(x, y) - x \cdot h(x, y)$.

4. Consider map of complexes given below. Compute the homology groups of each complex and then compute the induced map from the homology groups of the first complex to the homology of the second.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathbb{Z}^2 \arrow[r, "\left(\begin{smallmatrix} 9 & 9 \\ 6 & 6 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 6 \\ 4 \end{smallmatrix}\right)"] & \mathbb{Z}^2 \arrow[r, "\left(\begin{smallmatrix} -5 & 10 \\ 5 & -10 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 5 \\ -1 \end{smallmatrix}\right)"] & \mathbb{Z}^2 \arrow[r, "\left(\begin{smallmatrix} 10 & 6 \\ 5 & 3 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)"] & \mathbb{Z}^2 \arrow[r] \arrow[d, "\left(\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right)"] & 0 \arrow[d] \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot 6"] & \mathbb{Z} \arrow[r, "\cdot 0"] & \mathbb{Z} \arrow[r, "\cdot 16"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$