Goal: find a geometric interpretation of $Z_G(-)$ the TQFT corresponding to $Z\mathbb{C}G$, use our theory of characters to deduce nice geometric results.

We have an obvious basis for $Z\mathbb{C}G$ labelled by conjugacy classes $\{e_\alpha\}$ $\quad e_\alpha = \sum_{g\in\alpha}' g$

Tensor calculus: Given a basis for A we can express a linear map $f: A^{\otimes r} \to A^{\otimes s}$ via tensor coefficients:

$$f(e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_r}) = \sum_{\beta_1\cdots\beta_s}' f^{\beta_1\cdots\beta_s}_{\alpha_1\cdots\alpha_r}\, e_{\beta_1}\otimes\cdots\otimes e_{\beta_s}$$

we use <u>summation convention</u>: repeated indices, one up one down, are summed over $\quad f(e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_r}) = f^{\beta_1\cdots\beta_s}_{\alpha_1\cdots\alpha_r} e_{\beta_1}\otimes\cdots\otimes e_{\beta_s}$

f is determined by tensor coefs $f^{\beta_1\cdots\beta_s}_{\alpha_1\cdots\alpha_r}\in\mathbb{C}$

e.g. $f: A\to A \quad f^\beta_\alpha$ matrix entries.

Composition: $A\otimes A \xrightarrow{f} A\otimes A \xrightarrow{g} A$

$$e_\alpha\otimes e_\beta \longmapsto f^{\gamma\delta}_{\alpha\beta}\, e_\gamma\otimes e_\delta \longmapsto f^{\gamma\delta}_{\alpha\beta}\, g^{\varepsilon}_{\gamma\delta}\, e_\varepsilon$$

i.e. $(g\circ f)^\varepsilon_{\alpha\beta} = f^{\gamma\delta}_{\alpha\beta}\, g^\varepsilon_{\gamma\delta}$

Kronecker δ $\quad \delta^\beta_\alpha = \begin{cases} 1 & \alpha=\beta \\ 0 & \alpha\neq\beta\end{cases}$

so $f: A\to A$ has inverse $g \iff f^\alpha_\beta g^\beta_\delta = \delta^\alpha_\delta$

Non-degenerate form $g: A\otimes A \to \mathbb{C}$ given by $g_{\alpha\beta}$

Non-deg means induced map $A \to A^*$ is an iso.

The inverse map $A^* \to A$ corresponds to copairing $\mathbb{C} \to A\otimes A$ given by $g^{\alpha\beta}$; the inverse condition is $g^{\alpha\beta} g_{\beta\gamma} = \delta^{\alpha}_{\gamma}$.

The form g (or metric g) gives us a way of raising and lowering indices:

$$m: A\otimes A \to A \iff A\otimes A\otimes A^* \to \mathbb{C} \quad \text{then}$$

via $g: A^* \to A$ we get

$$A\otimes A\otimes A \to \mathbb{C}$$
$$v\otimes w\otimes u \mapsto g(m(v\otimes w), u)$$

in tensor language:

$m: A\otimes A \to A$ is given by $m^{\gamma}_{\alpha\beta}$

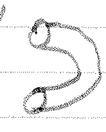
$A\otimes A\otimes A \to \mathbb{C}$ is given by $m_{\alpha\beta\gamma} := m^{\delta}_{\alpha\beta} g_{\gamma\delta}$

so for example co-multiplication $A \to A\otimes A$ is given by $m^{\varepsilon\eta}_{\delta} := m^{\gamma}_{\alpha\beta} g_{\gamma\delta} g^{\alpha\varepsilon} g^{\beta\eta}$

Any Frobenius alg. is determined by m, g, μ mult, pairing, counit.

$$A\otimes A\otimes A \xrightarrow{m} \mathbb{C} \qquad A\otimes A \xrightarrow{g} \mathbb{C} \qquad A \xrightarrow{\mu} \mathbb{C}$$

$m_{\alpha\beta\gamma}$, $g_{\alpha\beta}$, μ_{α}

$Z\Big($  $\Big)$ $\quad Z\Big($  $\Big)$ $\quad Z\Big($  $\Big)$

In the case of $\mathbb{Z}G$ $\{\alpha\}$ conj. classes of G

$$e_\alpha = \sum_{g \in \alpha} g$$

recall $\mu\left(\sum a(g) g \right) = \frac{1}{|G|} a(id)$

i.e. $\mu_\alpha = \mu(e_\alpha) = \begin{cases} \frac{1}{|G|} & \alpha = \{id\} \\ 0 & \alpha \neq \{id\} \end{cases}$

$$g\left(\sum a(g) g, \sum b(g) g \right) = \frac{1}{|G|} \sum a(g) b(g^{-1})$$

$g_{\alpha\beta} = g(e_\alpha, e_\beta) = \begin{cases} 0 & \alpha \neq \bar{\beta} \\ \frac{|\alpha|}{|G|} & \alpha = \bar{\beta} \end{cases}$ $\bar{\beta}$ conj class of inverse elts of β

let $Z(\alpha) = |\text{Centralizer}|$ $g_{\alpha\beta} = \frac{1}{Z(\alpha)} \delta_{\alpha\bar{\beta}}$

$g^{\alpha\beta} = Z(\alpha) \delta^{\alpha\bar{\beta}}$

$M_{\alpha\beta\gamma} = g(m(e_\alpha \otimes e_\beta) \otimes e_\gamma) = \langle e_\alpha \cdot e_\beta, e_\gamma \rangle = \langle e_\alpha \cdot e_\beta \cdot e_\gamma, 1 \rangle$

$= \mu(e_\alpha \cdot e_\beta \cdot e_\gamma) = \frac{1}{|G|} \cdot \left\{ \text{id. term of } \sum_{g \in \alpha} \sum_{h \in \beta} \sum_{k \in \gamma} ghk \right\}$

$M_{\alpha\beta\gamma} = \frac{1}{|G|} \# \left\{ g \in \alpha, h \in \beta, k \in \gamma : ghk = 1 \right\}$

Idea: make these numbers geometric - they should count something.

Jan 31st.

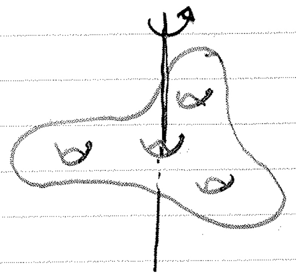
Def'n A free G-space is a topological space P on which G acts freely and continuously, i.e. for all $g \in G$

$$\phi_g : P \longrightarrow P \quad \text{continuous map.}$$

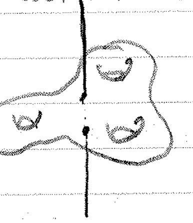
such that $\phi_{gh}(x) = \phi_g(\phi_h(x))$, $\phi_{id} = Id_X$

(free $x \neq \phi_g(x)$ unless $g = id$).

e.g. $\mathbb{Z}/3$ acts freely on:



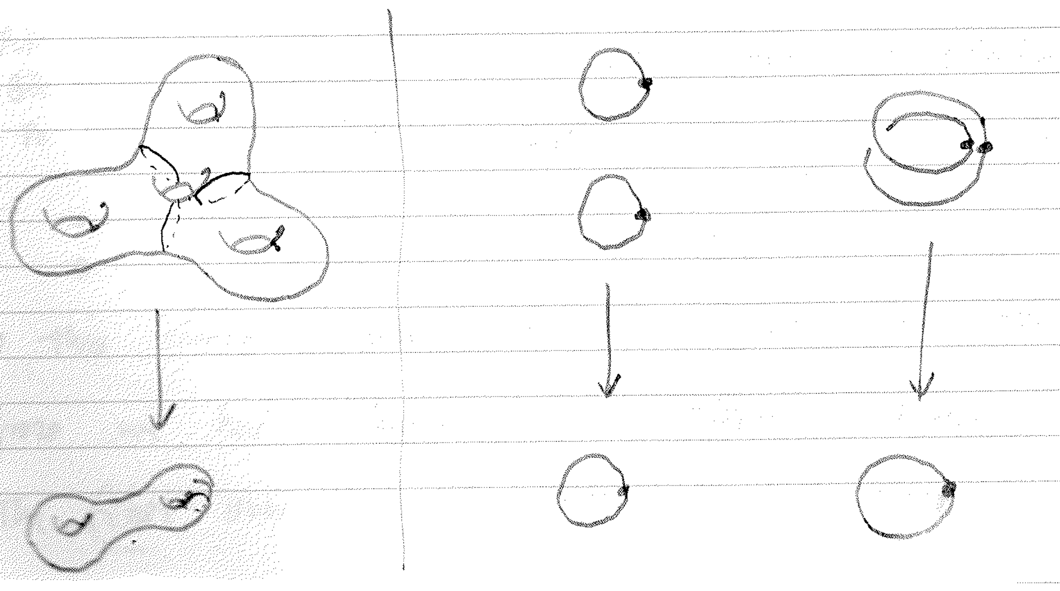
but similar action on



not free (2 fixed pts)

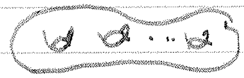
each orbit is a copy of G. Let $X = P/G$ be the orbit space: we identify $x \sim \phi_g(x)$

If P is a manifold then $X = P/G$ is a manifold



$\mathbb{Z}/2$ quotients

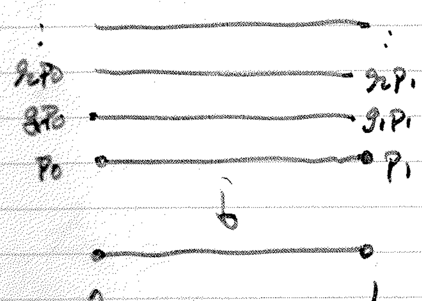
Def'n A principal G-bundle over X is a free G-space P such that $X = P/G$. $P = X \times G$ is called the trivial bundle.

Question: given a Riemann surface  X how many Principal G bundles over X are there?

(Related to Galois theory: If $P \to X$ is a pricipal G bundle then the field of meromorphic fncs on P is a Galois extension of the field of meromorphic fnc on X with galois gp G).

We begin by studying Principal G bundles over S^1.

We use the fact that all principal G bundles over $[0,1]$ are trivial. $S^1 = [0,1]/_{0\sim 1}$ so we can get $P \to S^1$ by gluing $G \times [0,1]$ to itself:



we can make a principal bundle on S^1 by gluing p_0 to $g p_1$ some $g \in G$

once this choice is made all other gluings are determined.

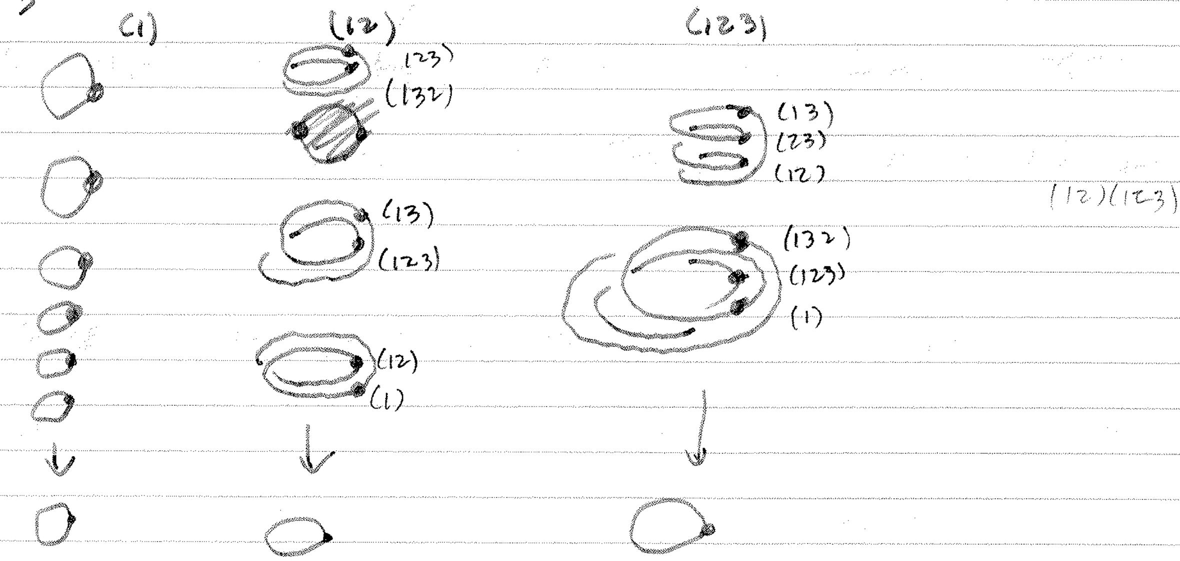
s'pose we had choosen a different labelling of the fiber over 0, i.e. some other $p_0' = h p_0$

then p_0' is glued to $hgh^{-1}p_1'$:

$$p_0' = hp_0 \sim hgp_1 = hgh^{-1}hp_1 = hgh^{-1}p_1'$$

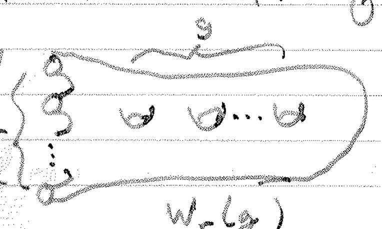
so the bundle is determined by a <u>conjugacy class</u>,
called <u>monodromy</u> (around the circle) of the bundle.

S_3 bundles on S^1 :



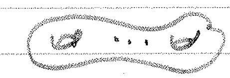
<u>Theorem</u> Let $Z_G(-)$ be the TQFT associated to the Frobenius alg. ZG. Let $Z_G(g)_{\alpha_1 \cdots \alpha_r}$ be the tensor coefs of

$$Z_G\left(\quad \right) : A^{\otimes r} \longrightarrow \mathbb{C}$$



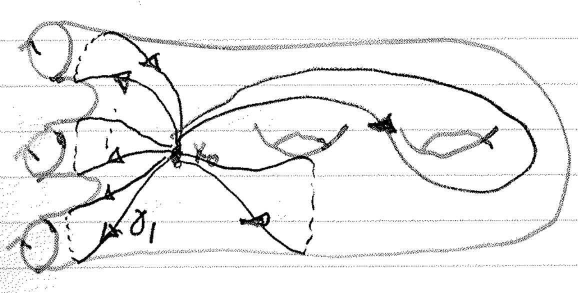
with respect to the basis $\{e_\alpha\}_{\alpha \text{ conj class of } G}$ $\quad e_\alpha = \sum_{g \in \alpha} g$

Then $Z(g)_{\alpha_1\cdots\alpha_r}$ = # of principal bundles on $W_r(g)$ having monodromy α_i about the ith boundary component. We count each bundle by $\frac{1}{\#\mathrm{Aut}\,P} = \frac{1}{|G|}$.

In particular $Z_G(g)$ = # princ. G-bundles on 

We sketch the proof. We show this forms a TQFT and we show it agrees with $Z\mathbb{C}G$ on pants, tube, and cap.

Pick a base point $x_0 \in W_r(g)$ and choose a point $p_0 \in \pi^{-1}(x_0)$. Then each loop $\gamma : [0,1] \to W_r(g)$ with $\gamma(0) = \gamma(1) = x_0$ determines an element of G by monodromy.



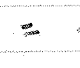
$\gamma_1, \dots, \gamma_r$ are loops homotopic to the loop around boundary components

This defines a homomorphism $\phi \in \mathrm{Hom}(\pi_1(W_r(g)), G)$ such that $\phi(\gamma_i) \in \alpha_i$ this uniquely determines the principal bundle. So

$\frac{1}{|G|}$ gets rid of choice of p_0

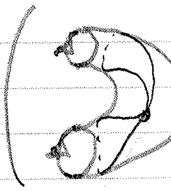
$$Z_G(W_r(g))_{\alpha_1\cdots\alpha_r} = \frac{1}{|G|} \# \{ \phi \in \mathrm{Hom}(\pi_1(W_r(g)), G) : \phi(\gamma_i) \in \alpha_i \}$$

$$Z_G(W_1(0))_\alpha = Z_G(\;\cdot\;)_\alpha = \frac{1}{|G|}\#\left\{\phi\in \mathrm{Hom}(1,G) : \phi\left(\begin{smallmatrix}\text{triv}\\ \text{loop}\end{smallmatrix}\right)\in\alpha_i\right\}$$



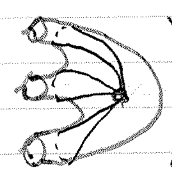
$$= \begin{cases} \frac{1}{|G|} & \alpha=\{id\} \\ 0 & \alpha\neq\{id\}\end{cases} \qquad = \mu(e_\alpha) \cancel{*} = \mu_\alpha$$

$$Z_G(W_2(0))_{\alpha\beta} = Z_G(\;\cdot\;)_{\alpha\beta} = \frac{1}{|G|}\cancel{\#}\left\{\phi\in \mathrm{Hom}(\mathbb{Z},G) : \begin{matrix}\phi(\gamma)\in\alpha \\ \phi(\gamma^{-1})\in\beta\end{matrix}\right\}$$



$$= \frac{|\alpha|}{|G|}\,\delta_{\alpha\bar\beta} \qquad = \langle e_\alpha, e_\beta\rangle = g_{\alpha\beta}$$

$$Z_G(W_3(0))_{\alpha\beta\gamma} = Z_G(\;\cdot\;)$$



$$= \frac{1}{|G|}\left\{\phi\in \mathrm{Hom}(\pi_1,G) : \phi(\gamma_1)\in\alpha \;\; \phi(\gamma_2)\in\beta \;\; \phi(\gamma_3)\in\gamma\right\}$$

free gp on two generators, or π_1 $\langle \gamma_1,\gamma_2,\gamma_3 : \gamma_1\gamma_2\gamma_3 = 1\rangle$

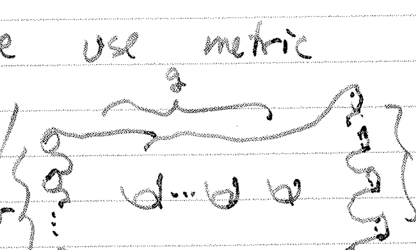
$$= \frac{1}{|G|}\left\{ \cancel{g_1 g_2 g_3}\; g_1\in\alpha,\ g_2\in\beta,\ g_3\in\gamma \;:\; g_1 g_2 g_3 = id\right\}$$

$$= \frac{1}{|G|}\ \text{Identity coef of } \sum_{g_1\in\alpha}\sum_{g_2\in\beta}\sum_{g_3\in\gamma} g_1 g_2 g_3 \quad = \langle e_\alpha\cdot e_\beta, e_\gamma\rangle$$

$$= m_{\alpha\beta\gamma}$$

To define whole TQFT we use metric

$$Z_G\Big(W_{r,s}(g)\Big)^{\beta_1\cdots\beta_s}_{\alpha_1\cdots\alpha_r} = Z_G(\;\cdot\;)^{\beta_1\cdots\beta_s}_{\alpha_1\cdots\alpha_r}$$

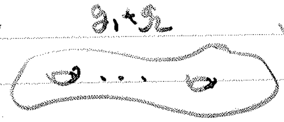
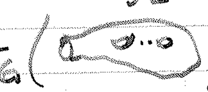


$z(\alpha) = |\text{centralizer}(\alpha)| = \frac{|G|}{|\alpha|}$

$:= z(\beta_1)\cdots z(\beta_s)\, Z_G(W_{g+s}(g))_{\alpha_1\cdots\alpha_r\bar\beta_1\cdots\bar\beta_s}$

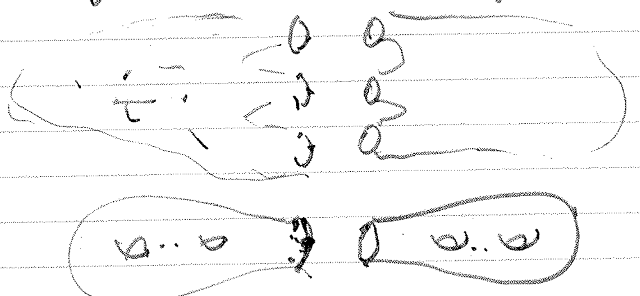
<u>Prop.</u> The above defines a TQFT, i.e. composition law * holds

e.g.

$Z_G\Big($  $\Big) \overset{?}{=} \sum_\alpha Z_G\Big($  $\Big)^\alpha Z_G\Big($  $\Big)_\alpha$

$= \frac{1}{|G|} \sum_\alpha |\alpha|\, Z_G(W_1(g_1))_{\bar\alpha}\, Z_G(W_1(g_2))_\alpha$

can glue bundles with opposite monodromy $\frac{|\alpha|}{|G|}$ number of ways to glue.



Since the principal bundle TQFT agrees with the one from ZCG on cap, pants, & tube they agree in general.

Our original topological problem:

$Z_G(g)$ = # Princ. G-bundles over closed genus g surface

$= \frac{1}{|G|} \# \mathrm{Hom}(\pi_1(\Sigma_g), G)$

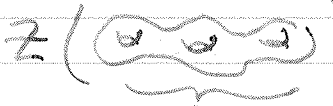
$= \frac{1}{|G|} \# \left\{ (a_1, b_1, \dots, a_g, b_g) \in G^{2g} : \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = 1 \right\}$

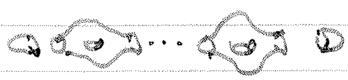
looks like a very difficult combinatorial group theory problem

Theorem $Z_G(g) = \sum_{\substack{\text{irr. repr} \\ R}}' \left(\frac{|G|}{\dim R} \right)^{2g-2}$

Def'n A TQFT / Frob alg is *semi-simple* if the alg. is semi-simple: i.e. $\exists$ a basis V_α s.t. $V_\alpha \cdot V_\beta = \delta_{\alpha\beta} V_\alpha$.

Exercise Suppose A is a semi-simple Frob. alg with idempotent basis $e_1, \dots, e_n$ (so $e_i \cdot e_j = \delta_{ij} e_i$) let $\mu_i = \mu(e_i)$. Show that $\mu_i \neq 0$ and letting $\lambda_i = \mu_i^{-1}$, show that

$Z\Big($  $\Big) = \sum_{i=1}^{n} \lambda_i^{g-1}$

Hint: compute , then 

Theorem ZCG is semi-simple with idempotent basis

$$V_R = \dim R \frac{1}{|G|} \sum_{g \in G}' \overline{\chi_R(g)}\, g$$

pf: For any repr. W and any irreducible R consider $\phi_R : W \longrightarrow W$

$$\phi_R(w) = \dim R \frac{1}{|G|} \sum_{g \in G} \overline{\chi_R(g)}\, g \cdot w$$

ϕ_R is G-linear:

$$\phi_R(hw) = \frac{\dim R}{|G|}\sum_{g\in G} \overline{\chi_R(g)}\,(hh^{-1})ghw \qquad g' = h^{-1}gh$$

$$= \frac{\dim R}{|G|}\sum_{g'\in G} \overline{\chi_R(g')}\, h g' w = h\,\phi_R(w)$$

By Schur's lemma $\phi_R : R' \to R'$ is equal to $\lambda\, Id_{R'}$ if R' is irr. and in this case

$$tr(\phi_R) = \lambda \dim R' = \dim R \frac{1}{|G|}\sum_{g\in G} \overline{\chi_R(g)}\,\chi_{R'}(g) = \delta_{RR'}\dim R$$

so $\lambda = \delta_{RR'}$ thus ϕ_R is projection onto R summands of W.

Let $W = R_{reg} = \mathbb{C}[G]$ then $\phi_R : \mathbb{C}[G] \to \mathbb{C}[G]$ is just multiplication by V_R and since $\phi_R \circ \phi_{R'} = \delta_{RR'}\phi_R$ we get $V_R \cdot V_{R'} = \delta_{RR'} V_R$ in $\mathbb{C}[G]$ since V_R are central and span $Z\mathbb{C}[G]$, the theorem follows.

$$\mu(V_R) = \mu\left(\dim R \frac{1}{|G|}\sum_{g\in G}\overline{\chi_R(g)}\,g\right) = \dim R \frac{1}{|G|}\cdot\frac{1}{|G|}\overline{\chi_R(id)}$$

$$= \frac{\dim^2 R}{|G|^2} \quad \text{so} \quad \lambda_R = \frac{|G|^2}{(\dim R)^2}$$

Exercise $\Rightarrow$ $Z_G(g) = \sum_R \left(\frac{|G|}{\dim R}\right)^{2g-2}$

verify formula for $g=0$ and $g=1$ "by hand"

$$Z(\,$$$$\,) = \frac{1}{|G|}\,\#\mathrm{Hom}(\underset{\{1\}}{\pi_1(S^2)}, G) = \frac{1}{|G|}$$ (the only bundle is the trivial bundle)

$$\overset{?}{=} \sum_R \frac{(\dim R)^2}{|G|^2} = \frac{1}{|G|^2}\,|G| = \frac{1}{|G|} \quad \checkmark$$

$$Z(\,$$$$\,) = \frac{1}{|G|}\,\#\mathrm{Hom}(\overset{\mathbb{Z}\times\mathbb{Z}}{\overset{\shortparallel}{\pi_1(\text{torus})}}, G))$$

$$= \frac{1}{|G|}\,\#\{a,b\in G \quad \underbrace{ab=ba}_{a=bab^{-1}}\}$$

$$= \frac{1}{|G|}\sum_{a\in G} |C(a)| = \sum_{a\in G} \frac{1}{|\mathrm{conj}(a)|}$$

$$= \sum_{\substack{\text{conj}\\ \text{classes}}} 1 = \#\text{ conj classes.}$$

$$\overset{?}{=} \sum_R \left(\frac{|G|}{\dim R}\right)^0 = \#\text{ irr repr} = \#\text{ conj. classes} \quad \checkmark$$

Relationship with covering spaces:

There is a correspondence:

$$\left\{\begin{matrix}\text{Principal } S_d\text{-bundles}\\ \text{on } X\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{degree } d\text{, not nec. connected,}\\ \text{covering spaces of } X\end{matrix}\right\}$$

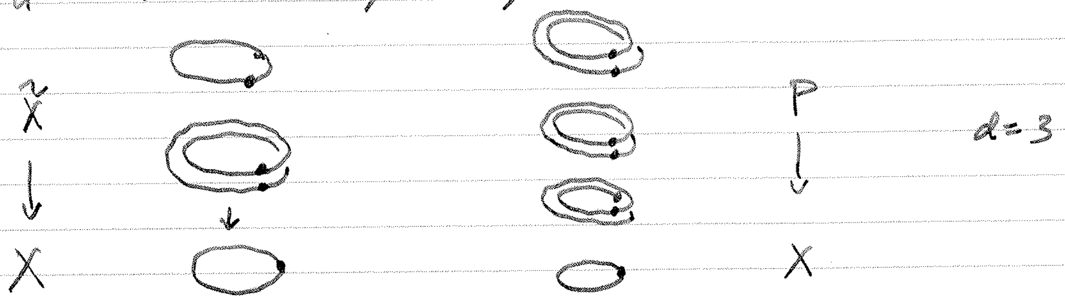
given a principal S_d-bundle $P \to X$ we can make a degree d cover $\tilde{X} \to X$ by $\tilde{X} = \left(P \times \{1,\dots,d\}\right)/S_d$

conversely, given a degree d cover $\tilde{X} \xrightarrow{\pi} X$ *

define $P = \{ \phi : \{1,\dots,d\} \hookrightarrow \tilde{X}$ s.t. $\pi \circ \phi$ maps to a point $\}$, i.e. labellings of the fibers $\tilde{X} \to X$

we must give P a topology so that the labellings vary continuously.

S_d acts on P by acting on $\{1,\dots,d\}$.



so

$$\#\ \text{deg } d \text{ covering spaces of } \Sigma_g = \sum_{\substack{\text{irr} \\ \text{rep of } S_d}} \left(\frac{d!}{\dim R}\right)^{2g-2}$$

(deg d covering spaces: not nec. conn.)

in particular $\#$ of deg d covers of $T^2 = p(d) = \#$ of partitions

very fun to see this directly from covering space theory:

$$\#\text{ of connected covering spaces of degree } d = \frac{1}{d}\ \#\text{ index } d \text{ subgps of } \mathbb{Z}\times\mathbb{Z}.$$

exercise: show RHS is $\frac{1}{d}\sigma(d)$ $\quad \sigma(d) = \#$ of divisors of d.

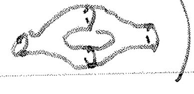
Let N_d = # of possibly disco covers = $p(d)$

let n_d = # of connected covers = $\frac{1}{d}\sigma(d)$

let $F(q) = \sum_{d=1}^{\infty} n_d q^d$

let $Z(q) = \sum_{d=0}^{\infty} N_d q^d$

show $Z(q) = \prod_{n=1}^{\infty} (1-q^n)^{-1} = \cancel{\log} \exp(F(q))$

Handle operator $H : Z\Big($  $\Big) : A \to A$

Let $\{\gamma_i\}$ be a basis for A

$H : \gamma_i \longmapsto m_i^{jk}\, \gamma_j \otimes \gamma_k \longmapsto m_i^{jk}\, m_{jk}^{n}\, \gamma_n$ *

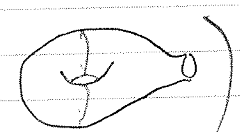
Let $\{\gamma^j\}$ be the dual basis, i.e. $\langle \gamma_i , \gamma^j \rangle = \delta_i^j$ $\quad \gamma^j = g^{jk} \gamma_k$

Let $\Gamma = \gamma_i \cdot \gamma^i \in A$

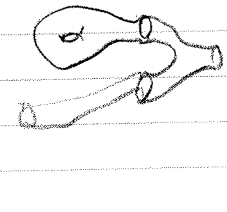
<u>Claim:</u> H is multiplication by Γ

<u>pf:</u>

$$
\begin{aligned}
\gamma_i \longmapsto\ & (\gamma_j \cdot \gamma^j) \cdot \gamma_i \\
&= \gamma_j\, g^{jk} \gamma_k \cdot \gamma_i \\
&= g^{jk} \gamma_j\, m_{ki}^{e}\, \gamma_e \\
&= g^{jk}\, m_{ki}^{e}\, m_{je}^{n}\, \gamma_n \\
&= g^{jk}\, g_{ka}\, m_i^{ea}\, m_{je}^{n}\, \gamma_n \\
&= m_i^{ej}\, m_{je}^{n}\, \gamma_n \qquad \blacksquare
\end{aligned}
$$

$Z\Big($  $\Big) : \mathbb{C} \longrightarrow A$

$\qquad 1 \longmapsto \Gamma$

 $\cong$