Homework problems  Math 502

1. Show that
\[ \chi_{\text{Sym}^2 V}(g) = \frac{1}{2} \left( \chi_V(g)^2 + \chi_V(g^2) \right), \]
\[ \chi_{\Lambda^2 V}(g) = \frac{1}{2} \left( \chi_V(g)^2 - \chi_V(g^2) \right). \]

2. The Character table of \( \mathbb{Z}/4\mathbb{Z} \). Write down the character table of \( \mathbb{Z}/4\mathbb{Z} \). Demonstrate the orthogonality of the characters of irreducible representations with three examples.

3. The Character table of \( D_8 \). Let \( S \) be the square in \( \mathbb{R}^2 \) with vertices \( x_1 = (-1, 1), x_2 = (1, 1), \)
\( x_3 = (1, -1), \) and \( x_4 = (-1, -1). \) The symmetry group of \( S \), is the group with eight elements
\[ D_8 = \{1, \tau, \tau^2, \tau^3, \sigma, \sigma \tau, \sigma \tau^2, \sigma \tau^3\}, \]
where \( \tau \) is given by counterclockwise rotation through 90 degrees and \( \sigma \) is reflection about the \( y \)-axis.

- Using the relations \( \sigma \tau \sigma = \tau^3, \sigma^2 = 1, \) and \( \tau^4 = 1, \) show that the five conjugacy classes of \( D_8 \)
are \( \{1\}, \{\tau, \tau^3\}, \{\tau^2\}, \{\sigma, \sigma \tau^2\}, \) and \( \{\sigma \tau, \sigma \tau^3\}. \)
- Let \( U \) be the trivial representation and let \( V \) be the complexification of the real two dimensional representation obtained from the above action on \( \mathbb{R}^2. \) Find the characters \( \chi_U, \chi_V, \) and \( \chi_{\Lambda^2 V}. \)
- Show that \( U, V, \) and \( \Lambda^2 V \) are irreducible and distinct.
- Use the properties of the character table to deduce the characters of the remaining two irreducible representations \( U' \) and \( U''. \)
- Find the decomposition into irreducible representations of the permutation representation of \( D_8 \) acting on the vertices \( \{x_1, x_2, x_3, x_4\}. \)

4. Symmetries of the Cube. Show that the permutation group \( S_4 \) is isomorphic to the group of rotational symmetries of the cube by considering the action on the four diagonals. Using the character table of \( S_4 \) from class, decompose the permutation representations given by the action of \( S_4 \) on

- The six faces,
- The eight vertices,
- The twelve edges.

Express in terms of the irreducible representations, \( U, V, U', V' = V \otimes U', \) and \( W. \)

5. The Character table of \( A_5. \) Consider \( A_5 \subset S_5 \) the alternating group.

- Show that \( A_5 \) has 5 conjugacy classes and give the number of elements in each.
• Show that the restrictions of the irreducible $S_5$ representations $U,V,W$ are irreducible and compute their character.

• Use the orthogonality of the character table to deduce the characters of the remaining two irreducible representations $Y$ and $Z$.

• Decompose $\Lambda^2 W$ into irreducible representations.